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OPERATIONAL CIRCUIT ANALYSIS

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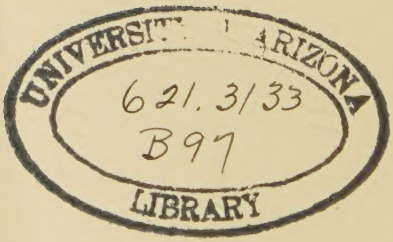

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PREFACE

There is now considerable literature on the subject of the operational analysis of circuits, which has developed and extended the methods introduced by Oliver Heaviside in his classic Electromagnetic Theory and Electrical Papers. Yet there is no text which gathers together the various parts of the subject in a form available to engineers and physicists. This the present treatment aims to accomplish, without undue emphasis on any one aspect or point of approach.

The operational method is very powerful, and it can be just as rigorous as the classic analysis on which it is based. It should be, and undoubtedly will be, increasingly used by all who deal with circuit transients. This contemplates not only the circuits of electricity, but also those of acoustics, mechanics, thermics, hydraulics, and so on. The object of this text is an exposition and substantiation of operational methods in a form applicable to all sorts of circuit problems. Illustrations are given freely and these are often electrical; but this is not a collection of interesting electrical problems, nor a treatment of transmission lines and cables.

It is entirely possible for a computer to perform the algebraic work necessary for the symbolic solution of alternating-current networks in the steady state without any grasp of the philosophy of the symbolic treatment or of the mathematics of differential equations on which it is based. The operational method bears much the same relation to transient problems that the symbolic method bears to steady-state problems. Both are shorthand processes based on classic circuit analysis. It is entirely possible to utilize the operational method on specific problems without the slightest idea of why and when it does or does not work. This, however, is computation and not analysis. In order to extend the operational method, to apply it freely and safely, to save

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time by the employment of its directness and simplicity on new problems, it is necessary to have at least some grasp of the classic mathematics for which it is the working tool. This means primarily a grasp of Fourier analysis and of its treatment in terms of the complex variable. Unfortunately this branch of mathematics is often absent from the mathematical equipment of engineers. Perhaps as good a way to approach the study of Fourier methods as any is through the operational scheme. I have therefore included a brief survey of certain essential parts of the classic treatment, although of course without any pretense to completeness in this regard. On the other hand I have attempted to include all the essential features of the Heaviside type of operational analysis, and to show their dependence upon the classic processes which constitute their background. This may make the reading difficult for those students who meet certain branches of mathematics for the first time; but it will serve a useful purpose if it demonstrates the usefulness of extending their mathematical equipment along the lines indicated. After all a little mathematical knowledge can be a dangerous thing, and the use of operational methods except for pure computation should be accompanied by an appreciation of the logic of more than simple algebra.

Finally it should be emphasized that I write as an engineer, and that I do not pretend to be a mathematician. I lean for support, and expect always so to lean, upon the mathematician, just as I must lean upon the chemist, the physician, or the lawyer. It is certainly incumbent upon me, however, to learn the language and as much as I can of the processes of thought of the mathematician, to do my part to bridge the chasm that so often separates the engineer and the pure scientist. I write in the hope that this will be the attitude also of those of my readers to whom this text will be useful. Perhaps it will not be uninteresting to the mathematician to note an engineer's approach to a problem with this situation in mind.

This work has resulted from a study made during several years of presenting the subject to graduate students (principally in electrical engineering) at the Massachusetts Institute of Technology. I hope that operational analysis will gradually become part of undergraduate study as well; that this text will help this to occur; and that it will be found useful in graduate courses in other fields than electrical engineering. There are many practising engineers who may find that it is suggestive or even of direct utility. With this in mind, I have included some problems that are rather more than simple exercises.

I have made many references in footnotes to publications dealing directly with operational methods or with allied matters. This is done not only to provide a guide to the literature of the subject, but also to acknowledge my indebtedness to the large number of workers in this field.

I am much indebted to many of my students and colleagues who have labored with me as this subject has so rapidly developed in the past decade. Professor M. S. Vallarta was perhaps the first to give freely of his inspiration and aid. He in fact wrote the first set of class notes which I used. I am much indebted also to Mr. M. F. Gardner, who has recently assisted me in the teaching of the subject, and who has read the manuscript and made suggestions, many of which I fear were sadly needed. Mr. Parry H. Moon and Mr. Nathan Howitt have also helped me enormously in corrections and with the proof, problems, and figures. I have made other acknowledgments in the text. Still more, however, I am indebted to Professor Norbert Wiener, of the Department of Mathematics of the Massachusetts Institute of Technology. He has patiently guided me around many a mathematical pitfall, and if I have still erred and fallen into some such, I wish to definitely go on record that it is not his fault. Moreover he has written an appendix to this text on certain mathematical points. I did not know an engineer and a mathematician could have such good times

together. I only wish that I could get the real vital grasp of the basic logic of mathematics that he has of the basic principles of physics. Finally I wish to express to Professor D. C. Jackson, the head of my department, and to President S. W. Stratton, of the Massachusetts Institute of Technology, my appreciation of the encouragement on their part which makes this sort of endeavor possible and pleasant.

V. Bush

Cambridge, Mass.
Jan., 1929.

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OPERATIONAL CIRCUIT ANALYSIS

CHAPTER I

THE CIRCUITS OF ENGINEERING AND PHYSICS

A surprising amount of engineering analysis is concerned with circuits of various sorts. Any one who can analyze circuits with facility under all sorts of conditions has mastered a very important part of engineering technique. This applies not alone to electrical engineering, where the universal use of circuits is strikingly apparent, but also to other branches; for there are circuits of hydraulics, mechanics, thermics, acoustics, and even chemistry. We evidently first need to define what is meant by a circuit, and to specify the properties of those which we shall here treat operationally.

A **circuit** may be defined as a physical entity in which varying magnitudes can be sufficiently specified in terms of time and a single dimension. It thus involves movement or variation along a path. This is contrasted with the **field** problem in which there is variation with two or three dimensions. Thus the analysis of the behavior of a speaking tube is a circuit problem, while a study of the production of sound in a room by a loud speaker is a field problem. In the speaking tube the air pressure and velocity may be assumed with sufficient engineering accuracy to be the same at a given instant for all points on a cross-section of the tube. It is hence possible to specify what is happening in the tube by specifying either the pressure or the velocity as a function of the time and a single dimension — the distance along the tube. In the loud-speaker problem, however, a description is necessary which involves every point of the room. This requires three dimensions, and a field problem is encountered.

Circuits may be either **closed** or **open.** With a closed circuit we have a cyclic space arrangement such that the path is continuous, and by proceeding along the circuit we can return to the starting point. A magnetic circuit is always closed for magnetic flux lines are always continuous. The path may vary in cross-section or in nature of material. Thus in the magnetic circuit of a dynamo one part may be the iron of the armature, another the air gap, and others the pole pieces and yoke; and the magnetic flux passes in a closed path through all these parts in series. As an example of an open circuit, consider a long copper rod one end of which is thrust into the fire. This is a thermal circuit, and the temperature and rate of heat flow are substantially functions of the single dimension of distance along the rod. Yet there is evidently no closed path and we have an open circuit.

A **network** is a combination of similar interrelated simple circuits. It is usual to speak of networks only when the interrelated circuits are of the same kind. Thus an electric bell connected to a dry battery gives an electric circuit, and two bells connected to the same battery give a network. But in the electric bell is a magnetic circuit necessary to its operation. The electric and magnetic circuits of the bell are interrelated, but the combination is not called a network. The number of degrees of freedom of a network is equal to the number of simple circuits composing it, or the minimum number of separate values of a variable necessary to specify its performance. The separate circuits of a network are often called **branches** or meshes. Usage is not fixed, and circuit is often used to mean a network with several branches.

2. **Circuit Problems and Field Problems.** Circuit analysis is perhaps the most important sort of technical analysis used by engineers, and especially by electrical engineers. Engineers have plenty of **field problems,** and would much like to be able to analyze them readily and completely. Take, for example, the problem in aeronautics of the flow of air past

airfoils, or the problems of ship resistance and wave formation, or the flow in steam or hydraulic turbines. Unfortunately it is usually possible to proceed only a short distance by formal analysis in problems such as these. Wherever the study can be reduced to involve one dimension only, it is much more amenable to analysis. Thus the hydraulic problem of the flow of water in a penstock and nozzle becomes manageable as a circuit problem, even for the transient conditions of flow involved in some water-power problems.

The evolution of electrical engineering has proceeded at such an extraordinary rate primarily because its technical problems have been such as to yield to analysis. This has made possible the exact design of apparatus and the accurate prediction of its performance. Fundamentally this has been true because most electrical engineering technical problems involve circuits, because these circuits can be accurately represented as such, and because the results of electric circuit analysis are readily checked by measurement. Electric machinery and other paraphernalia are essentially assemblages of interrelated circuits. The relationships involved electrically are very accurate, and it is much easier to insert an ammeter into an electric circuit than, for example, to put a flow meter into a pipe line.

For this reason circuit analysis has been developed primarily at the hands of electrical engineers, and the terminology of circuits is usually their terminology. This procedure will be continued in this text, but we shall keep in mind that other types of circuits are also important.

As electrical engineering develops, it becomes necessary to analyze new types of circuits. The circuit problems become more complex and inherently more difficult. Thus new methods of attack are necessary and are being developed, and relatively old methods are being polished into form for rapid use. The greatest tool for this purpose that is now being developed is the **Heaviside Operational Calculus.**

The present text is primarily devoted to the presentation of this tool and allied matters in form for engineering use. Operational methods have so far been applied only to circuit problems. There is no reason why they should not be very useful also in connection with field problems, but so far they have not been developed for this purpose.

Unless the power of engineering analysis keeps pace with the growing difficulty of its technical problems, rapid engineering progress will stop.

3. **Lumped and Distributed Parameters.** The characteristics of a circuit are specified in terms of circuit **parameters** or **constants.** The former is the preferable term, for sometimes the parameters are not fixed in value and it is confusing to call them constants. The inductance of an electric circuit, the weight of water in a unit length of pipe, the heat-storage capacity of a unit block of metal through which heat is flowing may all be circuit parameters.

Parameters may be either **lumped** or **distributed.** With a simple circuit containing only lumped parameters, one variable at least is a function of time only; that is, it has the same value at a given instant at all parts of a circuit. This is true for the flow of an incompressible fluid in an unyielding pipe. The parameters of circuits are never strictly lumped, but often it is sufficiently accurate to consider them so. Thus an inductance coil cannot be built without distributed capacitance, nor a condenser without a small amount of inductance in its leads.

When the parameters are distributed, there may be a space variation as well as a time variation of the variables in the circuit. Traveling waves can occur in the system. An electric transmission line, in which each element of the line may be assigned values of inductance and capacitance, may have at a certain instant many amperes at one point in the wire, and at the same instant no current at all in the same wire and only a few feet away. Suppose we pump water with a single-acting pump through a long length of very flexible

rubber tubing. The water will proceed in a series of pulses or waves, and at a given instant the flow of water at certain points may actually be backwards toward the pump.

The methods of analysis we shall consider may be applied both to circuits with lumped and with distributed parameters.

4. **Fixed and Variable Parameters.** The parameters of a circuit may be either **fixed** or **variable.** In case all the parameters of a circuit are fixed, the circuit is said to be **linear;** and, as we shall see, the principle of **superposition** may then be used. A fixed parameter is of course one which has the same value at all times, and irrespective of what is being done to the circuit at the moment. Circuit parameters are never absolutely fixed in any practical problem, but it is often very accurate to assume them so. It is usually quite justifiable to assume the resistance of a copper wire carrying a moderate current to be constant over a brief period of time. The same assumption in regard to the filament of an incandescent lamp may lead to gross errors.

Many parameters are substantially constant within certain limits. The leakage of an aerial transmission line is nearly zero until corona appears. When a steel rod is tapped on the end, the traveling sound waves in the metal may be analyzed by considering a circuit with fixed parameters, unless the elastic limit of the material is reached at some point.

When parameters vary, they may be functions of various things. The mutual inductance of a stationary and a rotating circuit in an alternator is a function of the time, and often closely a sinusoidal function of the time. The internal resistance of a rectifier is a function of the direction and amount of current through it. The resistance of a thermionic tube is ordinarily a function of both the grid and plate voltages, although in the shielded-grid tube it may be a function of the grid voltage only. With an iron-cored coil the inductance is strictly a function of the entire past history of the current, although often closely enough repre-

sentable as a function of the current only, or even as a constant.

The analysis of circuits with fixed parameters is much simpler than that of circuits with variable parameters. This text will not attempt any extended operational treatment of the latter, as operational methods have not as yet been successfully applied in practice to circuits with variable parameters. In Chapter XV, however, we shall outline the approach to the problem.

5. **Idealized Circuits.** All practical circuits must be idealized before they can be analyzed. Various approximations are introduced for simplification. The extent to which it is legitimate to proceed in this manner is dictated only by experience. Strictly, the accuracy of the mathematical solution of a physical problem can be determined only by experiment. Yet experience with similar problems will often tell us *à priori* about what approximations may be safely introduced.

Often a circuit problem is analyzed when actually a field problem exists. All electrical circuit problems are of this nature, although the approximation may be very close indeed. In analyzing power circuits at commercial frequencies we do not hesitate to neglect the fact that every alternating-current circuit radiates energy, for a simple computation will show that the amount of energy thus radiated is negligible. To do the same with the circuit of a short-wave radio receiver is to be wide of the mark. In fact, when the frequency is so high that the corresponding wave length is small compared to the physical dimensions of the apparatus, any treatment by means of circuits is utterly inaccurate. It is often customary, and in fact necessary if analysis is to proceed, to assume parameters fixed which are really variable, or lumped which are really distributed. Experience must be our guide in this matter. This emphasizes the fact that it is imperative for the engineer to see clearly the physical aspects of his analytical work at all

times. Every formula, every step should have for him a real and vital meaning in terms of copper and iron, the flow of water, or whatever he may treat.

Oftentimes there is a temptation for the engineer, dealing thus with approximations and guiding his every step with experiment, to be unduly careless of mathematical rigor. There is no excuse for carelessness in this respect any more than it is tolerable to use careless computations on important engineering works. Still it is not well to pay so much attention to questions of rigor that there is neither time nor effort left for anything else. Oliver Heaviside expressed himself so forcibly on this matter that further comment is hardly needed here, especially as it is improbable that any one will use this text who does not frequently turn for inspiration and background to Heaviside's own works, of which this is in some sense an interpretation.

PROBLEM ON CHAPTER I

Prob. 1–1. In the following cases note whether the parameters are actually fixed or variable, lumped or distributed. Where it would be reasonable to make assumptions, note what they are. Note whether a circuit problem or a field problem actually exists.

a. A switch is turned, lighting the headlights of an automobile. Find the current variation.

b. A steel ball is being heat treated, and is heated uniformly to a moderate temperature and then dropped into oil. Find the temperature distribution and variation.

c. The gates of a lock are suddenly opened when there is a difference of water level in the lock and the canal. Find the flow.

d. The suspension cable of a bridge is hung in place during erection with a large sag, and then slowly hauled in at one end. Find the stresses.

e During this process there is an accident, the cable slips back a few feet, and then is suddenly arrested. Find the stresses.

f. A large dish of water is evaporating in still air. Find the distribution of density of water vapor near the surface.

CHAPTER II

THE FUNDAMENTAL EQUATIONS AND CIRCUIT ANALOGIES

In analyzing circuits we seek to find the **effect** due to some known **cause,** or set of causes. In an electrical circuit the nature of the applied electromotive forces may be known and the resulting currents may be desired. In a hydraulic problem the pressures impressed on the system may be known, and an attempt made to find the flow of fluid. In a heat problem we are often acquainted with a given set of temperatures and search for the rates of heat transfer. For the present, we shall assume in each case that we know exactly how the cause varies with the time. Cases where the intensity of the cause varies with the amount of effect produced are difficult, but can often be treated by methods applicable to circuits with variable parameters.[1]

Sometimes the rôles played by the variables representing the usual cause and effect are interchanged, but the methods of analysis need be but little altered on that account. Thus, while it is usual to impress an electromotive force on an electrical circuit, it is sometimes desirable to impress a charge or a current. We are familiar with electric sources of substantially constant voltage, such as power mains and batteries, so that the use of these to cause current gives the usual case. The Thury system, however, is an illustration of a constant-current power source. With such a system, a motor is taken out of service by short-circuiting it and put into circuit by opening a switch across it. Thus a substantially constant current is suddenly impressed on the motor. This becomes the cause, and the resulting voltages are the effect.

[1] Chapter XV.

In heat-flow problems we often meet the case of a known impressed supply of heat producing temperatures for which we must analyze. Thus, when an electric flatiron is connected in circuit, the heat flow in calories per second is fairly constant and accurately known, and analysis will show the consequent distribution of temperatures in the metal.[1]

It makes little difference in method what type of circuit is analyzed or which variable is the cause. For simplicity of notation and language we shall usually treat the electric circuit, and consider as the cause a known set of applied electromotive forces which we can write as functions of the time. The method can then be carried over to other problems by analogy.

We shall thus write e for the applied electromotive force. This is a known function of the time:

$$e = f(t).$$

The same symbol may stand for an applied temperature or pressure difference. Similarly, i represents the current, but may also mean rate of heat flow, velocity of a fluid, etc. When usual cause and effect are reversed,

$$i = f(t)$$

is the known functional relation.

In a magnetic circuit, an applied magnetomotive force and a consequent magnetic flux must be considered. Since there is only a single parameter, reluctance, in a magnetic circuit, the entire performance is given by a law of the form of Ohm's law except where iron is present and the reluctance is variable. Special methods are then used, which need not be entered into here.

2. **Resistance.** The simplest circuit parameter is that which causes dissipation and which is called the **resistance.** It directly opposes the current in an electric circuit, con-

[1] This is discussed in greater detail in Chapter XIV.

verting electric energy into heat. It is defined by Ohm's law,

$$e = Ri \tag{1}$$

which states that for a simple resistance the ratio at any instant of voltage across it to current through it is given by the resistance. If the voltage is in volts and the current is in amperes, the resistance will be given in ohms.

In heat flow appears the exactly analogous thermal resistance. Its reciprocal is thermal conductance; and the thermal conductance of a unit cube is the thermal conductivity of the material. If in the above equation i represents the heat flow in calories per second and e is the temperature difference in degrees centigrade, R will be the thermal resistance as obtained from the thermal conductivity and the dimensions of the block of material, and will be measured in degrees per calorie per second.

In hydraulic systems, the parameter becomes fluid resistance. For flow through a pipe, R represents the pressure drop (in whatever units are being used) when the fluid is flowing steadily through the pipe at unit velocity.

In mechanical systems the analogous quantity is friction, and R will here represent the mechanical force necessary to maintain steady unit velocity.

When we write R we may mean any one of these parameters, but it will be simpler to confine ourselves to the single letter and to call it always the resistance. The same will be done with other parameters.

When the parameters of a circuit are lumped, R will represent the total resistance of a unit or element. With distributed parameters, we shall use the resistance of a unit length of the circuit and call this r. The same procedure will apply to the other parameters as well.[1]

3. **Inductance.** The **inertia** parameter of an electric circuit is its **inductance.** Just as resistance opposes a current, so inductance opposes a change of current. In other

[1] See Appendix A for a list of symbols used throughout this text.

words, a large inductance requires a relatively large voltage across it in order to produce a given rate of change of current through it. Inductance is defined by the relation:

$$e = L\frac{di}{dt} \tag{2}$$

which states that the voltage across a pure inductance is proportional to the rate of change of current through it, the proportionality factor being the value of the inductance. Inductance will be in henries when current is in amperes and potential difference is in volts. Of course either system of c. g. s. units may be used instead of the practical system.

In mechanical problems the inertia parameter is the **mass.** It is equal to the force required to produce unit rate of change of velocity — that is, unit acceleration. L may then stand for the mass in pounds or grams. As before, a small letter l will stand for inductance per unit length. In a fluid circuit, hydraulic or acoustic, l will be the mass of material in unit length of the circuit or pipe, and thus will be equal to the pressure difference per unit length due to accelerating the fluid at unit rate.

There is no detectable inertia parameter in heat flow. In the slow flow of a viscous fluid it may often be neglected. In electric cables, for slow rates of current change, it can also be left out of consideration in first approximations. These circuits are closely analogous.

4. **Capacitance.** The **spring** parameter of an electric circuit is its **capacitance.** Just as inductance opposes a change of current, so capacitance opposes a change of voltage. In other words, a large capacitance requires a relatively large current through it in order to produce a given rate of change of voltage across it. Capacitance is then defined by the equation:

$$i = C\frac{de}{dt} \tag{3}$$

An element has unit capacitance when unit rate of change of voltage across it corresponds to unit current through it. Since cause and effect are here reversed from the familiar order, we usually write

$$e = \frac{1}{C}\int i\,dt \tag{4}$$

or

$$e = \frac{q}{C} \tag{5}$$

and state that the capacitance of an element is given by the charge required to produce unit change of voltage across it. If the current is in amperes, the potential difference in volts, and the charge in coulombs, the capacitance will be in farads. By prefixing an "*ab*" to each of these units, we have the statement for the c. g. s. electromagnetic system; while by using a "*stat*," we get it for the c. g. s. electrostatic system of units.

The thermal capacitance is the charge in calories, or other heat unit, that must be communicated to the element to produce unit temperature change.

In fluid flow, the capacitance of an element is given by the quantity of fluid to produce unit pressure change. When the fluid is very compressible, as is air, the meaning is apparent; for it is easy to visualize the amount of air that must be pumped into an element such as a chamber in a pipe line to raise its pressure one unit, a pound per square inch, for example. The same holds for liquids also, but here the compressibility is so small that yielding of the walls of the containing vessel contributes largely to the total capacitance. The capacitance c per unit length of a hydraulic penstock may be readily computed from the known compression constants of water and the elastic properties of the pipe.

5. **Parameters of Mechanical Vibrating Systems.** The three parameters, R, L, and C are not the only parameters which may be set up to describe the properties of circuits.

However, we shall deal with circuits only when they can be specified completely in terms of these three. When the parameters are constant, there is no doubt of the convenience of this procedure.

Our definition of a circuit is rather broad and includes mechanical systems not usually regarded as circuits; but, on account of closeness of analogy, it is convenient to so treat them. Thus, a rigid pendulum is a system, the properties of which are readily described by these three parameters; and its motion is describable as a function of time and a single dimension. For a pendulum, e may represent an impressed torque, say in dyne-cm.; i, an angular velocity in radians per second; R, a damping torque in dyne-cm. per radian per second; L, a moment of inertia in gram-cm.^{2}; and C, a spring or gravitational factor in radians displacement per dyne-cm. restoring torque. All simple rigid mechanical oscillating systems of a single degree of freedom may be thus included in our treatment, together with coupled combinations of such systems. This includes such systems as oscillograph vibrators, galvanometers, telephone diaphragms, engine flywheels, vibrating shafts, and deflecting beams.

6. **Mutual Parameters.** So far we have dealt only with single circuits and the self-parameters of such circuits. In networks consisting of several circuits coupled together we have **mutual parameters.** Two circuits are coupled when they have an element in common, and this element may involve any one of the three parameters or any combination of them. Circuits will be denoted by Arabic numerals, as *1*, *2*, *3* . . . ; and parameters will be numbered with corresponding subscripts, as R_1, L_2, . . . (See Fig. 1). Mutual

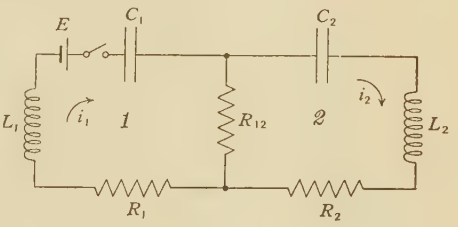

FIG. 1.

parameters will have two subscripts, as R_{12}, which shall be defined as the voltage produced in Circuit *1* when open by unit current in Circuit *2*.[1] When a circuit contains more than one parameter of the same kind, the total parameter of the circuit will be denoted by a double subscript, as R_{11}. ($R_{11} = R_1 + R_{12}$.)

In Fig. 1 is shown the simplest case — two electrical circuits with a **conductive coupling** by means of a simple resistance. The total self-resistance of Circuit *1* is equal to the voltage due to steady unit current in this circuit alone; that is, with no rate of change of current in the inductance and no charge on the condenser. This is evidently $R_{11} = R_1 + R_{12}$. Similarly, for Circuit *2* we have $R_{22} = R_2 + R_{12}$. The mutual resistance, R_{12}, is equal to the voltage produced in *1* when steady unit current flows in *2*, Circuit *1* being open. The same result is obtained for the voltage in *2* when unit steady current flows in *1*. We shall call this R_{21}. The relation is reciprocal, and we can express the fact thus:

$$R_{12} = R_{21} \tag{6}$$

The same holds true for other parameters. The **coefficient of coupling** is defined as the mutual resistance divided by the geometric mean of the self-resistances, or

$$\frac{R_{12}}{\sqrt{R_{11}\,R_{22}}} \tag{7}$$

When the entire resistance is common, the coefficient of coupling is unity. The same terminology holds for coupling through other parameters.

[1] A uniform system of double-subscript notation will be used throughout this book, not only for the circuit parameters R_{hk}, L_{hk}, C_{hk}, z_{hk}, but also for the minor M_{hk}, for the impedance function Z_{hk}, for the indicial admittance A_{hk}, and for the indicial impedance, V_{hk}. In all cases we shall observe the convention that the first subscript, h, shall refer to the circuit in which voltages are being considered, while the second, k, shall refer to the circuit in which currents are being considered. In other words, h is related to a voltage; k, to a current.

With a common inductance, as in Fig. 2, we have **inductive coupling.** Evidently $L_{11} = L_1 + L_{12}$, etc. as before.

Inductive coupling is possible when two coils have a common magnetic flux even when there is no actual wire

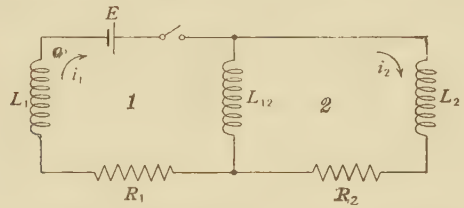

FIG. 2. Inductively Coupled Network.

common to the two circuits. (Fig. 3.) The definitions still hold.

The self-inductance of Circuit *1* is equal to the voltage resulting from unit rate of change of current in *1*, with Circuit *2* open or absent, and the same for the other circuit. The mutual inductance is the voltage in *1* resulting from unit rate of change of current in *2*, or *vice versa*. In a practical

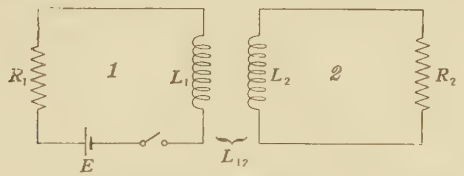

FIG. 3. Circuits with Common Magnetic Linkages.

case these constants are of course determined by examination of the design or by actual test. The mutual inductance will be denoted by L_{12}.

We can also have **condensive coupling,** as, for example, in Fig. 4. The self-capacitance of Circuit *1* is the value of charge we must pass through *1* to produce unit voltage in Circuit *1*, Circuit *2* being open. This is seen to be

$$C_{11} = C_1 + \frac{C_2 C' C''}{C_2(C' + C'') + C'C''}. \quad \text{Similarly for } C_{22}. \quad (8)$$

The mutual capacitance is the charge which must be passed through *1* to produce unit voltage in *2*, or *vice versa*. This is

$$C_{12} = C_1 + C_2 + C_1C_2\left(\frac{1}{C'} + \frac{1}{C''}\right) \tag{9}$$

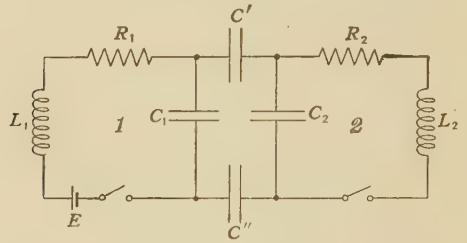

FIG. 4. Network with Condensive Coupling.

It is also possible to have condensive coupling between circuits when they have a common electrostatic field, and without any metallic connection between them. In Fig. 5 the circles may represent either spheres or cross-sections of parallel cylindrical conductors. The self and mutual ca-

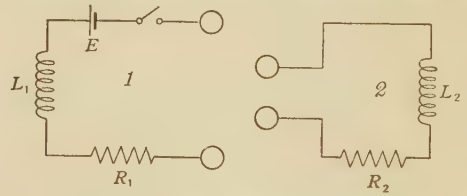

FIG. 5. Circuits with Common Electrostatic Field.

pacitances are defined exactly as before, the potentials being computed from the configuration of the electrostatic field.

In the above figures we have shown in each case only two circuits with a single common parameter. There may be many individual circuits or **branches** in a network, and the branch between any two may contain any or all of the three types of coupling.

It is also possible to have networks of other sorts of circuits than electrical. In an alternator there are as many magnetic circuits as there are poles, and they are coupled by a com-

mon reluctance in parts of each circuit. Usually, due to symmetry, it is only necessary to compute one circuit.

If a spring is connected between two pendulums we have a condensively-coupled network. The mutual capacitance is then the angular deflection of one which will produce unit torque on the other.

Air pipes and chambers may be interconnected to form an acoustic network. Acoustic filters, which are selective to frequency, have been built in this manner. The analysis of such systems is analogous to that of electric filters.[1]

Interconnected thermal circuits may also be considered.

In all these cases the analysis proceeds in exactly the same manner, the interpretation of the equations being clear if the definition on which they are set up is clear.

7. **Fundamental Circuit Laws. Kirchhoff's Laws** enable us to write the circuit equations. The first law states that the sum of all the voltages about a closed circuit is zero. This follows directly from the conservation of energy. All voltages must be included: applied electromotive forces, voltages due to the effect of self-parameters, and those induced by coupling with neighboring circuits.

The second law states that the sum of the currents flowing into a junction point is zero, and follows from the conservation of electricity. We shall consider the positive direction of current in each circuit as clockwise. In a common element the second law then enables the resulting current to be written with due regard to sign.

8. **Operational Terminology.** We shall use **operational notation** from the outset. Instead of

$$e = L\frac{di}{dt} \tag{10}$$

let

$$e = Lpi \tag{11}$$

[1] See, for instance, G. W. Stewart, Acoustic Wave Filters, Phys. Rev., **20,** 1922, p. 528; **23,** 1924, p. 520; **25,** 1925, p. 90.

W. P. Mason, A Study of the Regular Combination of Acoustic Elements, B.S.T.J., **6,** 1927, p. 258.

where p is simply another way of writing $\frac{d}{dt}$. There is nothing new in this, of course.

We are quite used to **operators** and **operands.** An operator is a symbol which states that a certain operation is to be performed on the quantity to which it applies and which is called the operand. Thus, in the expression,

$$y = \cos x$$

the **cos** is a symbol for a certain familiar operation, and x is the operand. The matter would be much simpler if our notation were consistent, but, like many similar things which just grew, it is far from it. In the expression,

$$y = \log x$$

the operator and operand are clear. But in the inverse expression,

$$x = \epsilon^y$$

it is not nearly so evident that we denote an operation on y.

Often a unit negative exponent means the inverse operation. Thus, in

$$x = \cos^{-1} y$$

the $\cos^{-1}$ is the inverse operator to **cos**, meaning that when they are applied together their effects annul. Thus,

$$x = \cos^{-1} \cos x = \cos \cos^{-1} x.$$

But while in the expression,

$$x^2$$

the exponent 2 means to perform the operation of squaring,

$$x^{-2}$$

denotes the reciprocal, not the inverse, which latter is written:

$$x^{\frac{1}{2}}.$$

We do not get confused by these things because of long familiarity, however much we may have struggled in early

days. It is unfortunate that operational notation is so complicated; but we are not responsible for it, and the best we can do is to try to be simple and consistent in any new operators we may introduce.

In setting up the differential equations of a network, we deal with the **derivative operator** p, and various combinations of such operators which are always applied to continuous functions of time.[1] p is here simply an abbreviation for $\frac{d}{dt}$; and, when applied to a continuous function of the time, it takes the derivative or slope of that function. When the operational solution of networks is dealt with, we shall

[1] A large amount of work has been done on operators and operational methods. We shall refer to much of this literature at points where it particularly applies or in order to acknowledge the source of material included in the text. At this point we note a few papers which give widely different methods of approach to the subject, to some of which we shall again refer later on.

Bernstein and Doetsch, Probleme aus der Theorie der Wärmeleitung, Math. Zeits., **22**, 1925, p. 284.

T. J. I'A. Bromwich, Examples of Operational Methods in Mathematical Physics, Phil. Mag., **37**, 1919, p. 407.

G. C. Evans, Functionals and their Applications, Am. Math. Soc., Cambridge Colloquium, Vol. 5, 1918; Discontinuous Dirichlet and Neumann Problems, Am. Math. Soc., Colloquium Publications, **6**, 1927.

P. Lévy, Leçons d'Analyse Fonctionnelle, Gauthier, Paris, 1922.

E. Picard, Sur la Distribution de l'Électricité avec la Loi de Neumann et sur la Theorie analytique de la Chaleur. Rendiconti del Circolo Matematico Palermo, **37**, 1914, p. 249.

Pleijel and Liljeblad, Operatorkalkylens Samband med den Symboliska Metoden, Teknisk Tids., 1919, p. 25.

J. B. Pomey, Le Calcul Symbolique d'Heaviside, Rev. Gén. d'Élec., **13**, 1923, pp. 813, 860.

H. Salinger, Die Heavisidesche Operatorenrechnung, E.N.T., **2**, 1925, p. 365.

J. J. Smith, The Solution of Differential Equations by a Method Similar to Heaviside's, Jour. Franklin Inst., **195**, 1923, p. 815; An Analogy between Pure Mathematics and Operational Mathematics of Heaviside by Means of the Theory of H-Functions, Jour. Franklin Inst., **200**, 1925, pp. 519, 635, 775.

introduce boundary conditions, and shall find our operators acting on discontinuous functions. The operator p is then no longer interpretable simply as a derivative operator. We shall, in fact, interpret it as we evaluate it; so that our procedure at first is heuristic or experimental, and definitions will follow later. These definitions and laws governing operators might be set up in various ways and still lead to the correct solutions of problems. The desideratum is of course to set up a system which is as simple and convenient as possible, and which will yield correct results. As we introduce operational processes we shall continually bear this in mind. The **inverse operator** will be written p^{-1}. p^{-1} is then defined as that operator which annuls the effect of p, so that

$$p^{-1}pf(t) = f(t). \tag{12}$$

The operation p^{-1} is thus an integration, since it cancels the effect of a differentiation. Just what sort of integration it is depends on circumstances, as will be shown later.

All of algebra depends upon the **associative, distributive,** and **commutative laws,** which may be grouped and called the **algebraic laws.** If a, b, and c are algebraic quantities, these laws state that

$$\begin{aligned} a(bc) &= (ab)c \\ a(b + c) &= ab + ac \\ ab &= ba. \end{aligned} \tag{13}$$

When quantities obey these laws, and only then, all the usual algebraic transformations may be applied.

In the theory of differential equations it is shown that the operator p obeys the laws of algebra when it is applied to functions that are continuous.

Many operators are used in mathematics which do not obey these laws, and which cannot, therefore, be combined by the usual algebraic rules. In establishing our operational processes we shall at first simply assume that the operators

we utilize obey these laws, and later we shall justify this assumption.

Using the exponent notation,

$$ppf(t) = p^2 f(t)$$
$$p^m p^n f(t) = p^{m+n} f(t), \text{ etc.}$$

We shall use the functional notation, often using fairly complicated combinations of p with various constants. $F(p)$ may mean such an expression containing p. Various operators may be distinguished by subscripts. When an operator $F(p)$ is applied to a time function $f(t)$ it generates another time function. Such a function, $F(p)$, is therefore called a **generating function.**

Note one special point: the expression,

$$F_1(p)f_1(t) \cdot F_2(p)f_2(t)$$

is equal to

$$F_2(p)f_2(t) \cdot F_1(p)f_1(t)$$

for it is a product of two time functions. The order cannot be changed in any other way without altering the meaning.

9. **Differential Equations of Lumped Networks.** For a single circuit with fixed lumped parameters, not coupled to other circuits, Kirchhoff's first law states that

$$R_1 i_1 + L_1 p i_1 + \frac{1}{C_1 p} i_1 = e_1. \tag{14}$$

The successive terms are the voltages produced by i_1 in conjunction with resistance, inductance, and capacitance; while e is the applied electromotive force which is assumed to be a known function of the time:

$$e_1 = f_1(t).$$

If this is abbreviated by writing

$$R_1 + L_1 p + \frac{1}{C_1 p} = z_{11}(p) \tag{15}$$

the equation becomes

$$z_{11}(p) i_1 = f_1(t).$$

If this circuit be coupled to a second circuit in all three ways, new terms must be added, giving

$$z_{11}(p)i_1 + R_{12}i_2 + L_{12}pi_2 + \frac{1}{C_{12}p}\,i_2 = f_1(t).$$

Abbreviating as before,

$$z_{11}(p)i_1 + z_{12}(p)i_2 = f_1(t). \tag{16}$$

If there are n circuits in the network, there will be n such terms, and a similar equation may be written for each circuit of the network. The **complete differential equations** for a network of n circuits containing lumped fixed parameters are then

$$\left.\begin{aligned} z_{11}(p)i_1 + z_{12}(p)i_2 + \cdots z_{1n}(p)i_n &= f_1(t) \\ z_{21}(p)i_1 + z_{22}(p)i_2 + \cdots z_{2n}(p)i_n &= f_2(t) \\ \cdots\cdots\cdots\

times carelessly used when parameters are not fixed, thus producing some rather startling results.

A network with fixed parameters is called a **linear network,** and the equations which describe it are known as linear differential equations. The theory of such equations shows why **superposition** is legitimate with a linear network. The reason can also be visualized physically. If a parameter always has the same value, no matter what may be occurring in the circuit at the time, then a single impressed force will arouse the same reaction no matter what other forces may be acting simultaneously. Thus the result of several forces acting simultaneously is found by adding the effects which would be produced by each acting alone.

In the differential equations of a linear network this means that there is no loss of generality if only one impressed force is considered, noting that if there are several, we shall be able to thus treat each in turn for a complete result.

Therefore,

$$\left.\begin{aligned} z_{11}(p)i_1 + z_{12}(p)i_2 + \cdots + z_{1n}(p)i_n &= f_1(t) \\ z_{21}(p)i_1 + z_{22}(p)i_2 + \cdots + z_{2n}(p)i_n &= 0 \\ \cdots\cdots\cdots\

The signs are in accordance with the convention of positive clockwise current in each circuit. They are best kept straight by noting the relative way in which the current flows through mutual elements. If i_2 had been taken counterclockwise, all

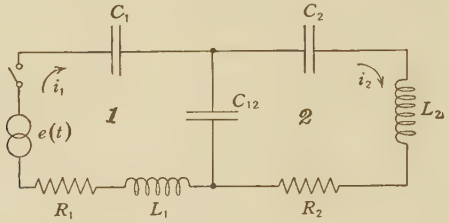

FIG. 6.

signs would have been plus. It makes no difference in the final result which system is adopted.

When two circuits are inductively coupled by a transformer, there is an opportunity for confusion in signs. Especially is this true when three circuits are thus coupled in pairs by three transformers. It then pays to actually trace out the connection of the windings and note the direction of the voltages induced when the current in each circuit in turn has a positive rate of increase.

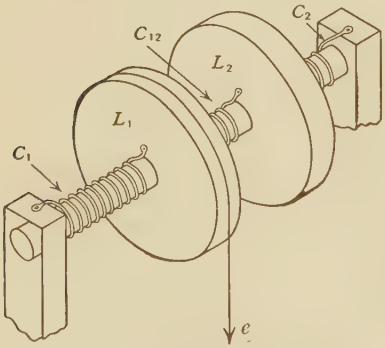

FIG. 7. Mechanical Coupled Circuits.

The above circuit equations hold exactly as written for the mechanical system of Fig. 7. The flywheels are loose on the shaft; and each is attached to a helical spring, the other end of which is fixed. A third spring connects the two flywheels. Any known torque variation may be considered applied at e, and this is $f(t)$. L_1 and L_2 are the moments of

inertia of the two flywheels. R_1 and R_2 are the frictional torques of each at unit angular velocity. C_1, C_2, and C_{12} are the spring constants of the three springs; that is, the deflection in radians to produce unit restoring torque. i_1 and i_2 are the angular velocities of the two wheels. The equations assume that all parameters are fixed, which may be seriously in error with regard to the frictional terms.

In Fig. 8 is shown a rather artificial hydraulic network which has the same equations. L_1 and L_2 are the total moving masses in the two pipes, and R_1 and R_2, the corresponding fluid friction parameters. The applied force $f(t)$ is exerted on the piston. The capacitances correspond to the spring constants of the three diaphragms. Voltages correspond to pressures, and currents to rates of fluid flow.

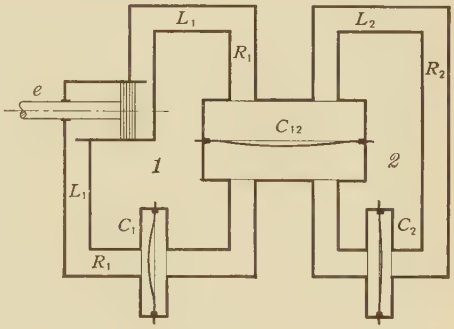

FIG. 8. Hydraulic Coupled Circuits.

The only safe way to set up analogies between different sorts of circuits is to actually write the differential equations and identify the meaning of each term.[1] This also serves to keep the matter of units clear.

[1] For a treatment of a practical application of mechanical analogues, see Maxfield and Harrison, Methods of High Quality Recording, Trans. A.I.E.E., **45**, 1926, p. 334; B.S.T.J., **5**, 1926, p. 493.

C. R. Hanna, Loud Speakers of High Capacity, Trans. A.I.E.E., **47**, 1928, p. 607.

V. Bush, Gimbal Stabilization, J. Franklin Inst., Aug. 1919.

C. A. Nickle, Oscillographic Solution of Electro-Mechanical Systems, Trans. A.I.E.E., **44**, 1925, p. 844.

PROBLEMS ON CHAPTER II

Prob. 1–2. Draw a network consisting of two circuits coupled together inductively, conductively, and electrostatically. Make the network as simple as is possible under these conditions.

Prob. 2–2. A crude seismograph is built by attaching an iron ball to a rod, mounting the rod horizontally so that its end pivots against the mopboard of a wall, and then supporting the whole by a wire fastened to the rod near the ball and running at an angle of 45° to a screw eye in the wall in a vertical plane through the center of the rod. A smoked plate is arranged to detect any relative motion of the ball and the floor. An earthquake moves the building as a whole, and from the record produced it is desired to show what was the component of earth motion perpendicular to the rod. Write the differential equations. What is the analogous electric circuit problem? Identify each parameter, and note its exact definition.

Prob. 3–2. A radioactive substance A changes into a second substance B, which in turn changes into C, which is a permanent element. The rate at which A transforms is proportional to the amount of A present, and the same is true for B. Required, the laws of change. Write the differential equations. Set up an analogous problem in hydraulics. Also discuss the electric circuit analogue.

Prob. 4–2. For which of the following electric circuits will the principle of superposition hold strictly, or approximately, and within what limits?

a. A direct-current generator supplying the filament of an incandescent lamp.

b. A filter constructed of air-core coils and paper condensers, for use at audio frequencies.

c. A telephone circuit consisting of microphone, battery, line, and receiver.

d. A 220-kv. power transmission line with its terminal transformers.

e. An arc-type electric furnace.

Prob. 5–2. A telephone line with wires spaced one foot horizontally runs for a mile parallel to a power transmission line. The three wires of the power line are spaced 6 feet in an equilateral triangle, the base of which is horizontal and 30 feet high. The telephone loop is 10 feet high, and its center line is 12 feet horizontally from the center line of the power line. During trouble on the power line a large single-phase current flows in the two wires nearest the telephone line, the current in the third wire being negligible. It is required to find the effect produced by interference on the telephone line, when the latter is terminated at the ends of the mile exposure by being (*a*) open, (*b*) short cir-

cuited. What parameters are involved, and how would they be computed?

Prob. 6–2. A network is made up of three circuits, each containing all three parameters, and every circuit being inductively coupled to each of the other two. Set up the differential equations, with particular regard to signs.

Prob. 7–2. A pendulum is hanging at rest from a support which is suddenly displaced a short distance horizontally. What is the analogous problem of electric circuits?

Prob. 8–2. Two pendulum bobs are coupled by means of a spring having a stiffness factor $\left(\frac{1}{c}\right)$ of 200 dyne-cm/radian. The masses of the pendulums are 100 g. and 300 g. respectively, while their lengths are 100 cm. and 50 cm. A steady force of 100 dynes is suddenly applied to the 100-gram bob in such a direction as to tend to separate the two pendulums. Neglect friction and also the mass of the bob suspensions. Write the differential equations and give the analogous electric circuit.

Prob. 9–2. Derive equations (8) and (9).

Prob. 10–2. Write the differential equations for the circuits of Figs. 1, 2, 3, and 4 using the p-notation.

Prob. 11–2. Sketch a mechanical analogue of Fig. 4.

Prob. 12–2. The usual type of low-pass electric filter consists of a system of series inductances and shunt capacities. Write the differential equations for such a filter of four sections terminated in a resistance R. Sketch a mechanical analogue such that when a sinusoidal translatory motion of sufficiently low frequency is applied to the input, a similar motion of nearly the same amplitude appears at the output.

Prob. 13–2. An oscillating torque which is periodic but not sinusoidal is applied to a shaft. The power is taken off a second shaft having the same center line as the first. Devise a mechanical filter, of three T-sections, to connect the two shafts and to eliminate all frequency components above 200 cycles/sec. The cut-off frequency is $1/(\pi\sqrt{lc})$.

Prob. 14–2. One type of band-pass filter has capacitances for shunt elements, while its series elements consist of capacitances and inductances in series. It is proposed to make a "hydraulic band-pass filter" to be used in a submarine signal system to eliminate sound waves of too high or too low a frequency. What might be its construction?

Prob. 15–2. An oscillograph vibrator is a highly-damped pendulum. The deflecting force on the vibrator is proportional to the current through it. What are the parameters of the vibrator system? What is the electric-circuit analogue?

Prob. 16–2. A crystal, in the form of a thin layer the area of which is large compared to its thickness, is dropped into pure water and dissolved. Considering this as a problem of pure diffusion and neglecting effects near the edge of the crystal, what constants enter into the formulation of the problem, and what is the analogous electric circuit?

CHAPTER III

THE IMPEDANCE FUNCTION

To analyze for the behavior of a network means to solve the differential equations of the network; that is, to find expressions for the various currents which, substituted into these equations, will satisfy the initial and final conditions, and the terminal conditions if there are distributed parameters present.

We seek, then, expressions for the current which, substituted into the left-hand members, will give the right-hand members. To such expressions could evidently be added expressions which, substituted into the left-hand members, would give zero; and the sum of these expressions would still satisfy the equations. The solution of a set of linear differential equations thus falls into two parts: the first is the **particular integral,** and the second, the **complementary function.** The solution of a set of differential equations can also be divided into the **steady-state term** and the **transient term:** a part which is in form like the applied force, and a part which approaches zero as t increases. Ordinarily these parts are obtained on solving for a particular integral and a complementary function. We deal at present with the steady-state term: that is, the series of current expressions which will give the impressed forces when substituted into the left-hand side of the equations.

2. **Response to Exponential Electromotive Force.** From a mathematical standpoint, the simplest form of electromotive force which can be applied to a network is one which varies exponentially. This is because of the simplicity of the derivative and the integral of an exponential. If a current of the form:

$$i = I\epsilon^{bt}$$

flows through a resistance R, the voltage resulting is

$$e = Ri.$$

If it flows through an inductance L, the voltage is

$$e = Lpi = LpI\epsilon^{bt} = LbI\epsilon^{bt} = Lbi.$$

If it flows through a capacitance C, the voltage is

$$e = \frac{1}{Cp}i = \frac{1}{Cp}I\epsilon^{bt} = \frac{1}{Cb}I\epsilon^{bt} = \frac{1}{Cb}i.$$

When an exponential electromotive force is applied to a linear network, the expression for the resulting current in each branch will contain an exponential steady-state term of the same form. It must evidently be of this form if it is to yield an expression equal at all times to the applied e.m.f., for the form of an exponential does not change when differentiated or integrated. If an electromotive force,

$$e = E_1\epsilon^{bt}$$

be applied to Circuit *1* of a network, the resulting steady-state currents may be written

$$i_1 = I_1\epsilon^{bt}$$
$$i_2 = I_2\epsilon^{bt}$$
$$\text{etc.}$$

where I_1, I_2, etc. are constants we wish to evaluate.

Substitute these expressions for the currents into the general equations and write $z(b)$ for $R + Lb + \frac{1}{Cb}$, obtaining

$$\left.\begin{aligned} &z_{11}(b)I_1\epsilon^{bt} + z_{12}(b)I_2\epsilon^{bt} + \cdots + z_{1n}(b)I_n\epsilon^{bt} = E_1\epsilon^{bt} \\ &z_{21}(b)I_1\epsilon^{bt} + z_{22}(b)I_2\epsilon^{bt} + \cdots + z_{2n}(b)I_n\epsilon^{bt} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &z_{n1}(b)I_1\epsilon^{bt} + z_{n2}(b)I_2\epsilon^{bt} + \cdots + z_{nn}(b)I_n\epsilon^{bt} = 0 \end{aligned}\right\} \quad (21)$$

If the exponential which occurs in each term is eliminated, we have left simply a set of algebraic equations containing

the values of I:

$$\left.\begin{aligned} z_{11}(b)I_1 + z_{12}(b)I_2 + \cdots + z_{1n}(b)I_n &= E_1 \\ z_{21}(b)I_1 + z_{22}(b)I_2 + \cdots + z_{2n}(b)I_n &= 0 \\ \cdots\cdots\cdots\

equally easy to solve for the steady-state response to an alternating applied e.m.f. This is the **symbolic** method of solving alternating-current circuits.

Consider that in Circuit *1* there is applied the electromotive force,

$$e = E_1 \cos \omega t.$$

This is identical with[1]

$$e = \frac{E_1}{2} \epsilon^{j\omega t} + \frac{E_1}{2} \epsilon^{-j\omega t}.$$

The applied e.m.f. can now be considered in two parts; and, the circuit being linear, they can be applied separately and the results added. Since each term is exponential, the results of the preceding section may be used directly, with $j\omega$ in place of b.

The loop current in Circuit k due to the first term is

$$\frac{E_1}{2\, Z_{1k}(j\omega)} \epsilon^{j\omega t} \tag{29}$$

and due to the second term,

$$\frac{E_1}{2\, Z_{1k}(-j\omega)} \epsilon^{-j\omega t}. \tag{30}$$

An examination of the determinant shows that $Z(j\omega)$ and $Z(-j\omega)$ are conjugate complex quantities. Thus,

$$\left.\begin{aligned} Z_{1k}(j\omega) &= |\, Z_{1k}(j\omega)\,| \angle \phi = |\, Z_{1k}(j\omega)\,| \, \epsilon^{j\phi} \\ Z_{1k}(-j\omega) &= |\, Z_{1k}(j\omega)\,| \angle -\phi = |\, Z_{1k}(j\omega)\,| \, \epsilon^{-j\phi} \end{aligned}\right\} \tag{31}$$

so the total loop current in Circuit k is

$$I_k = \frac{E_1}{2\,|\, Z_{1k}(j\omega)\,|} (\epsilon^{j\omega t} \epsilon^{-j\phi} + \epsilon^{-j\omega t} \epsilon^{j\phi})$$

$$I_k = \frac{E_1}{|\, Z_{1k}(j\omega)\,|} \cos (\omega t - \phi). \tag{32}$$

[1] We here use $j \equiv \sqrt{-1}$, since in electrical problems it is desirable to reserve i for current. In mathematical texts, however, i is usually used for the pure imaginary.

We have here carried through the entire process. Usually it is abbreviated and only one of the terms used, a convention being introduced so that the j is removed from the final expression. When both terms are used it disappears automatically as it necessarily must, for the currents that flow are real, not imaginary. Many a student of electrical engineering has wondered what is meant by an imaginary expression for a real current.

The $Z(j\omega)$ above is the **alternating-current impedance** of the network. It may be formed from the determinants as above, or often more readily by combining the impedances of elements in series and parallel in accordance with methods very familiar to electrical engineers.

$Z_{11}(j\omega)$ is the impedance of the network as examined from a point in Circuit *1*. When divided into the e.m.f. applied to this circuit, it will give the loop current which flows in this circuit; its magnitude giving the magnitude of current and its angle giving the phase angle between current and voltage.

$Z_{1k}(j\omega)$ is a **transfer impedance.** Divided into an electromotive force applied in Circuit *1*, it gives the loop current, which will flow in Circuit k.

$$Z_{1k}(j\omega) = Z_{k1}(j\omega).$$

From this follows the **reciprocal theorem:** If a given e.m.f. in one branch produces a certain current in a second branch, then with the conditions reversed and the given e.m.f. applied in the second branch, the same current as before will flow in the first branch. In other words, if the applied e.m.f. remains unchanged, the source of e.m.f. and the current-measuring instrument may be interchanged in a network without altering the current reading. This theorem will be extended below.

4. **Importance of the Impedance Function.** The impedance Z of a network completely specifies it electrically, provided the network has fixed parameters. If there are only two terminals of the network available, then the Z

at this point contains all possible information concerning the network. If there are two pairs of terminals available, the Z at each point and the transfer impedance between them gives the information.

Evidently, in accordance with the preceding section, if two networks have the same Z-function at some point, they will respond identically to the same alternating voltage applied at this point, no matter what may be their internal connections. A similar statement may be readily made when several pairs of terminals are available and all the Z's are identical. It is not so evident, but is just as true as we shall see later, that the two networks will respond in identical manner[1] to a voltage of any form whatever applied to their terminals. In other words, two networks which have the same Z at available terminals cannot be distinguished by any electrical tests whatever so long as all measurements are confined to conditions at these terminals; assuming, of course, that the applied voltages are kept within bounds so that all networks remain fixed. As an example, consider the three very dissimilar networks shown in Fig. 9. The Z-function of each is simply equal to R. If all three were in boxes with only the terminals a–b available, it would be impossible to tell them apart by electrical tests only.[2]

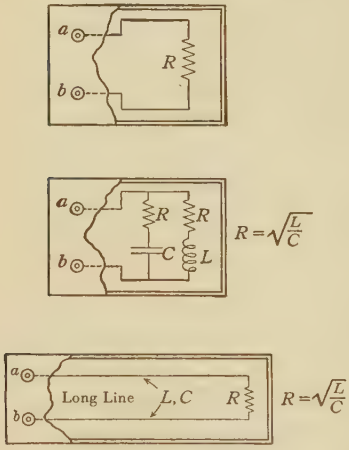

FIG. 9. Electrically Equivalent Networks.

[1] With identical transient currents as well as identical steady-state currents.

[2] See J. Slepian, Discussion of Murnaghan's paper, Trans. A.I.E.E., **47**, p. 369, 1928.

It is evident that the impedance function Z is an important characteristic of a network. It is readily formed, and **it contains within itself all the information necessary for the determination of the circuit performance under any conditions whatever.** Unfortunately, the relationships by which it will yield this information are not always simple.

5. **Impedance of Distributed Circuit.** So far only circuits with lumped parameters have been treated. The impedance function can also be readily obtained for **circuits with distributed parameters.** With distributed circuits, a partial differential equation takes the place of a set of simultaneous total differential equations. In order to find the impedance function, we find the ratio between the steady-state voltage and current for a sinusoidal, or simpler still, for an exponential voltage. We shall do so for the most usual type of distributed circuit as represented by the electric transmission line. The same treatment applies to hydraulic or acoustic pipe lines or to mechanical vibrating systems in which the parameters are distributed.[1] In the line of Fig. 10 let the parameters be r ohms, l henries, c farads, each per unit length of line. There may be also leakage between lines given by a conductance of g mhos per unit length.

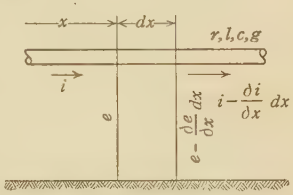

FIG. 10. Element of a Transmission Line.

Consider a differential length dx of line. The potentials at the two ends of this element of length differ by an amount given by the ohmic and inductive drop in the element, or

$$\frac{\partial e}{\partial x}dx = -ri\,dx - l\,dx\frac{\partial i}{\partial t} \qquad (33)$$

where the minus signs are due to the convention that the positive direction of both i and x is to the right. In this

[1] See, for instance, Maxfield and Harrison, *loc. cit.*

expression, e and i mean the voltage and current at a point x units from the generator end. Similarly, the currents at the two ends differ by reason of current which leaks between lines and current which flows into the element capacitance. Thus,

$$\frac{\partial i}{\partial x}dx = -ge\,dx - c\,dx\frac{\partial e}{\partial t}. \tag{34}$$

These two equations:

$$\left.\begin{aligned}\frac{\partial e}{\partial x} &= -ri - l\frac{\partial i}{\partial t}\\ \frac{\partial i}{\partial x} &= -ge - c\frac{\partial e}{\partial t}\end{aligned}\right\} \tag{35}$$

are the partial differential equations of the system.

If currents are now considered to be of the exponential form, then due to the properties of the exponential, steady-state voltages will be of the same exponential form, or

$$\left.\begin{aligned}i &= I\epsilon^{bt}\\ e &= E\epsilon^{bt}\end{aligned}\right\} \tag{36}$$

where I and E are functions of x alone.

Our equations then become total differential equations; the time disappears when the exponential is eliminated.

$$\left.\begin{aligned}\frac{dE}{dx} &= -rI - lbI\\ \frac{dI}{dx} &= -gE - cbE.\end{aligned}\right\} \tag{37}$$

Eliminating between these,

$$\left.\begin{aligned}\frac{d^2E}{dx^2} &= (r + lb)\,(g + cb)\,E\\ \frac{d^2I}{dx^2} &= (r + lb)\,(g + cb)\,I.\end{aligned}\right\} \tag{38}$$

Abbreviating with

$$(r + lb)\,(g + cb) = \beta^2 \tag{39}$$

we have simply

$$\left.\begin{aligned}\frac{d^2E}{dx^2} &= \beta^2 E\\ \frac{d^2I}{dx^2} &= \beta^2 I.\end{aligned}\right\} \tag{40}$$

Solutions of these are

$$\left.\begin{aligned}E &= A\epsilon^{\beta x} + B\epsilon^{-\beta x}\\ I &= C\epsilon^{\beta x} + D\epsilon^{-\beta x}.\end{aligned}\right\} \tag{41}$$

By noting that

$$\frac{dI}{dx} = -(g + cb)\,E \tag{42}$$

for all values of x, we can reduce the constants of integration to two, and write

$$\left.\begin{aligned}E &= A\epsilon^{\beta x} + B\epsilon^{-\beta x}\\ I &= -A\frac{g + cb}{\beta}\epsilon^{\beta x} + B\frac{g + cb}{\beta}\epsilon^{-\beta x}.\end{aligned}\right\} \tag{43}$$

With the further abbreviation:

$$z_0 = \sqrt{\frac{r + lb}{g + cb}} \tag{44}$$

these are simply

$$\left.\begin{aligned}E &= A\epsilon^{\beta x} + B\epsilon^{-\beta x}\\ I &= -\frac{A}{z_0}\epsilon^{\beta x} + \frac{B}{z_0}\epsilon^{-\beta x}.\end{aligned}\right\} \tag{45}$$

Now the line may be terminated in various ways at its ends. As a special case, suppose that it is supplied with voltage at one end and extends to an infinite distance. It is evident physically that if x increases without limit, the voltage and current cannot rise without limit. Hence A in the above must be zero.

$$\left.\begin{aligned}E &= B\epsilon^{-\beta x}\\ I &= \frac{B}{z_0}\epsilon^{-\beta x}.\end{aligned}\right\} \tag{46}$$

The ratio of voltage to current at any point of the line is then simply equal to z_0. The impedance function for a long line of this sort, examined at any point, is then

$$Z(b) = \sqrt{\frac{r + lb}{g + cb}}. \tag{47}$$

The alternating-current impedance is

$$Z(j\omega) = \sqrt{\frac{r + jl\omega}{g + jc\omega}}. \tag{48}$$

The impedance function for various other terminations of the line is similarly obtained and will be found in texts on transmission lines.[1] We need review only this one case to illustrate the treatment of the partial differential equation.

6. **Steady-state Analysis.** The general problem of the steady-state response of linear networks to alternating impressed electromotive forces is of great importance to engineers, but cannot be entered into here. This text deals rather with the transient behavior of networks, and more particularly with the analysis of circuit transients by operational processes, especially those of Heaviside. Our interest in the impedance function comes from the fact that it is the most convenient way of summing up the electrical characteristics of the network.

There is a considerable literature on the steady-state performance of networks. For the distributed circuits of power transmission, the usual vector diagrams which are so useful in steady-state analysis have been developed into circle diagrams which summarize an enormous amount of information in small compass.[2] Networks in which the same circuit is repeated several times and which exhibit strong frequency selective properties are called filters; and their

[1] For example, see L. F. Woodruff, Principles of Electric Power Transmission, John Wiley & Sons, 1925, p. 133.

[2] See Woodruff, *loc. cit.*, Chapter XII, or E. A. Loew, Electrical Power Transmission, McGraw-Hill, 1928, Chapters XIV, XV.

steady-state treatment is found in a large number of articles and texts.[1] Certain special networks, such as the transformer, have treatments especially adapted to them. In all these cases the treatment can be applied to non-electrical networks by analogy.[2]

PROBLEMS ON CHAPTER III

Prob. 1-3. Solve for the steady-state currents in both inductances of Fig. 1, using determinants and minors. Consider the impressed voltage to be $Ee^{j\omega t}$.

Prob. 2-3. A network of three circuits has all three parameters in each circuit. Circuits *1* and *2* are coupled by a common condenser, while Circuits *2* and *3* are coupled by a common resistance. Find the impedance function for a point in Circuit *1:* (*a*) by combining impedances in series and parallel, (*b*) by using determinants.

Repeat (*a*) and (*b*) for the transfer impedance Z_{13}.

Prob. 3-3. A long copper rod is to be heated at one end, and the flow of heat studied. Set up the differential equations, and form the impedance function. (This will be discussed in Chapter XI, but may be considered here by analogy with the long transmission line, noting what parameters are negligible.)

Prob. 4-3. A thin-walled rubber tube is filled with water, closed at one end, and connected to a reciprocating pump at the other end. Note the parameters of the tube, and its impedance function. A somewhat similar circuit appears in considering the flow of blood in arteries.

[1] *Cf.* G. W. Pierce, Electric Oscillations and Electric Waves, McGraw-Hill, 1920, Chapter XVI.

G. A. Campbell, Physical Theory of the Electric Wave-Filter, B.S.T.J., **1,** 1922, #2, p. 1.

L. J. Peters, Theory of Electric Wave-Filters Built up of Coupled Circuit Elements, Trans. A.I.E.E., **42,** 1923, p. 98.

O. J. Zobel, Theory and Design of Uniform and Composite Wave-Filters, B.S.T.J., **2,** 1923, p. 1; Transmission Characteristics of Electric Wave-Filters, B.S.T.J., **3,** 1924, p. 567.

K. S. Johnson, Transmission Circuits for Telephonic Communication, Van Nostrand, 1925.

Johnson and Shea, Mutual Inductance in Wave Filters, B.S.T.J., **4,** 1925, p. 52.

[2] Notably in the case of acoustic filters. Consult the papers of G. W. Stewart, *loc. cit.* or that of W. P. Mason, *loc. cit.*

Prob. 5–3. Two power transmission lines built in identical manner run out 20 and 50 miles respectively from a power station. They are connected to the same bus bars at the station; and terminate at the distant end in impedances σ_1 and σ_2 respectively. Set up the expression for the impedance function of each line alone, and for the total impedance measured at the bus bars. Formulate the transfer impedance for one line between bus bars and load. See texts on transmission lines (*e.g.*, Woodruff, Principles of Electric Power Transmission and Distribution).

Prob. 6–3. The impedance function of a lumped network may be readily found by the usual rules for combining the alternating-current impedances of the elements in series and parallel. This will always work except in the case where a portion of the network appears in a form similar to a Wheatstone bridge. Then a delta-star transformation of three elements will always eliminate the difficulty. This matter should be reviewed in any good text on alternating-current circuits (*e.g.*, Lawrence, Principles of Alternating Currents). Satisfy yourself that this procedure will yield both driving-point and transfer impedance functions by setting up various lumped networks and outlining the procedure of making the combinations. Transformers should first be replaced by their equivalent circuits, and the same with electrostatic couplings when such couplings are not simply made up of condensers.

Prob. 7–3. Determine the velocity in the mechanical resistance of Prob. 12–2 when the applied force is $E\epsilon^{j\omega t}$ dynes. What is the transfer impedance between the first and last circuits?

Prob. 8–3. Solve for the steady state of i_2 in the analogous electric circuit of Prob. 8–2, using an applied force of 100 sin 6 πt.

Prob. 9–3. A loud speaker having an inductance of 1 h. and resistance of 1000 ohms is connected in series with a capacitance of 2 mfd. and the combination is shunted by a 30-henry choke of 500-ohms resistance. To this parallel group is connected an alternating voltage, E sin ωt, through a 2000-ohm resistance. In this amplifier-output device the 2000 ohms and the voltage E sin ωt represent approximately the plate circuit of the last triode of an amplifier. Plot the velocity of the speaker diaphragm as a function of frequency between 30 and 5000 cycles per sec. with constant alternating voltage, E sin ωt. In what frequency range would you consider this output device satisfactory? Assume that the diaphragm displacement is at all times directly proportional to the current through the loud-speaker winding, calling the velocity unity at 1000 cycles/sec.

CHAPTER IV

THE INDICIAL ADMITTANCE

The transient analysis of networks can be best based upon the response to a suddenly-impressed unit voltage. Such a voltage may be impressed on a network by closing a switch and thus connecting a battery of unit e.m.f., or it may be impressed by suddenly inserting into the closed network a source of one volt. In either case the voltage to which the network is subjected is zero up to a certain instant, and unity thereafter. This fundamental sort of impressed voltage shall be called a unit voltage.

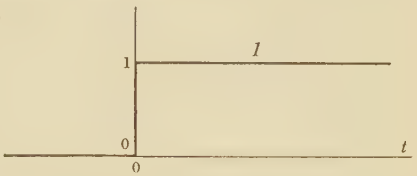

FIG. 11. The Unit Function.

More generally, we shall specify by **unit function** that discontinuous function[1] of the time which is zero until t equals zero and unity thereafter. It is plotted in Fig. 11. Following Heaviside, we shall denote such a function by the symbol *1*. This unusual way of writing a time function is convenient because it can also be used in connection with other functions as a brief way of denoting discontinuity. Thus ϵ^{at} represents the usual exponential function of the time, but $\epsilon^{at}\ 1$ represents that function of time shown in Fig. 12,

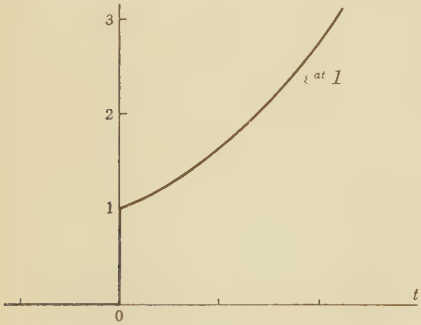

FIG. 12. Discontinuous Exponential Function.

[1] In mathematical texts this function is often written $\frac{1}{2}(1 + sgn\ t)$.

which is zero until t equals zero and exponential thereafter. The expression is the product of two time functions — an exponential, and a unit function — but the *1* can also be regarded as a symbol calling attention to the fact that the quantity to which it is appended has a discontinuity at the origin.

The unit function is important, for it is the simplest possible discontinuous function. The transient analysis of networks is the study of their behavior when subjected to discontinuities. The simplest discontinuity has a fundamental significance, for all transient behavior can be expressed in terms of it.

2. **Definition of Indicial Admittance.** The response of a network to unit e.m.f. (*1*) is called the **indicial admittance** of the network, and is denoted by $A(t)$.[1] $A(t)$ is thus the current which flows in the network when unit e.m.f. is suddenly applied, it being assumed throughout that the network is initially at rest with zero current and voltage everywhere. $A(t)$ thus gives the fundamental direct-current transient of the network.

The current that will flow in Circuit k when unit electromotive force is applied to Circuit h is denoted by

$$A_{hk}(t)$$

and is called the **transfer indicial admittance** between Circuits h and k.

The indicial admittance of a network completely characterizes the network. This same statement was made in regard to the impedance function. There must, therefore, be a definite relation between the two; so that if Z for a network is known, A for the same network can be found,

[1] The term "indicial admittance" was introduced by J. R. Carson, and has been generally adopted. For graphs of this function for a number of different circuits, see Carson, Trans. A.I.E.E., 1919, or Carson and Zobel, Transient Oscillations in Electric Wave-Filters, B.S.T.J., July, 1923, p. 1.

and *vice versa*. In fact there are several such relations which are mathematically identical when used under conditions to which they all apply, for they express the same interdependence but in widely different forms. A considerable portion of this text is devoted to their development and use.

3. **The Steady-state Solution.** The indicial admittance of a network can of course be found by classic methods; that is, the differential equations can be solved and the constants evaluated by means of the initial conditions which correspond to the application of the unit electromotive force.

When unit voltage is applied to Circuit h, the current in Circuit k will be given by a steady-state term and a transient term. The steady-state term for i_k is simply

$$\frac{1}{Z_{hk}(0)}.$$

This can be taken from our previous solution for the steady-state response to an applied voltage of the exponential form by putting $b = 0$, which reduces the exponential to a unit voltage.

The value of $Z(0)$ for a network — that is, the impedance of the network at zero frequency — is, of course, nothing but the direct-current resistance. This can be found most readily by inspection, ignoring all inductances and condensers in the circuit and combining the individual resistances in series and in parallel. In this process the inductances are considered of zero impedance and the condensers of infinite impedance.

4. **Classic Treatment of Transient Term.** In order to obtain the transient term, we need to find the complementary function which satisfies the equations:

$$\left.\begin{aligned} z_{11}(p)i_1 + z_{12}(p)i_2 + \cdots + z_{1n}(p)i_n &= 0 \\ z_{21}(p)i_1 + z_{22}(p)i_2 + \cdots + z_{2n}(p)i_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ z_{n1}(p)i_1 + z_{n2}(p)i_2 + \cdots + z_{nn}(p)i_n &= 0. \end{aligned}\right\} \qquad (49)$$

Let us assume that a solution of these equations exists in the form:

$$\left.\begin{aligned} i_1 &= K_{11}\epsilon^{\lambda t}, \\ i_2 &= K_{22}\epsilon^{\lambda t}, \text{ etc.} \end{aligned}\right\} \tag{50}$$

If we substitute these into (49) and divide out the exponential, the following algebraic relations are obtained:

$$\left.\begin{aligned} &z_{11}(\lambda)K_{11} + z_{12}(\lambda)K_{22} + \cdots + z_{1n}(\lambda)K_{nn} = 0 \\ &z_{21}(\lambda)K_{11} + z_{22}(\lambda)K_{22} + \cdots + z_{2n}(\lambda)K_{nn} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &z_{n1}(\lambda)K_{11} + z_{n2}(\lambda)K_{22} + \cdots + z_{nn}(\lambda)K_{nn} = 0. \end{aligned}\right\} \tag{51}$$

Such a set of algebraic equations has a solution only when the determinant of the system vanishes; that is, when

$$D(\lambda) = 0.$$

D has the same meaning as in Chapter III. This is an algebraic equation in λ. In general it will have $2n$ roots:

$$\lambda_1,\ \lambda_2,\ \lambda_3, \ldots \lambda_{2n}$$

where n is the number of circuits in the network, although it may have less than this number if some parameters are absent. The equation $D(\lambda) = 0$ is called the **determinantal equation** of the network.

Expressions of the form:

$$i_k = K_{k1}\epsilon^{\lambda_1 t} + K_{k2}\epsilon^{\lambda_2 t} + \cdots (2n \text{ terms}) \tag{52}$$

therefore satisfy our differential equations, as may be verified by substitution. Thus we have the form of the transient portion of our solution, although the values of the constants of integration K_{k1}, K_{k2} . . . are as yet undetermined.[1]

[1] Note that the double subscript used here with K identifies the particular current and the particular root to which that value of K belongs.

The expressions of the form $Ke^{\lambda_k t}$ are called the **normal modes** of the network, and the values λ_1, λ_2 . . . are called the **generalized natural angular velocities,** for reasons that will appear.

5. **Determination of Generalized Angular Velocities.** Since we have

$$Z(\lambda) = \frac{D(\lambda)}{M(\lambda)} \tag{53}$$

we shall in general obtain the same roots, or the same values of λ_1, λ_2 . . . , if we equate $Z(\lambda)$ to zero as when we equate $D(\lambda)$ to zero. For the present it shall be assumed that all the roots are different. The case of coincident roots will be discussed later.[1] We shall also scrutinize cases in which a different set of roots will be obtained from Z than from D.

One way of finding the natural angular velocities of a network is hence to form the alternating-current impedance of the network:

$$Z(j\omega)$$

in any convenient manner, replace $j\omega$ by λ, and solve

$$Z(\lambda) = 0.$$

Usually in a transient solution a term will appear corresponding to each value of λ obtained as a root. There are special conditions under which some of these terms may not appear. This will be probed in Chapter VII.

In general the same values of λ will be obtained on equating $Z(\lambda)$ to zero, no matter which point in the network is considered in forming Z. Evidently we shall always get the same roots as from $D(\lambda)$ unless one of these roots happens also to make $M(\lambda)$ zero. This special case also requires some discussion later. For the present, however, it will be assumed that Z may be formed looking into the network from any convenient point. This corresponds to the fact that in general the same modes of natural oscillation will

[1] See p. 91.

appear in a network, no matter how it is jarred out of equilibrium. The exceptions to this rule require special treatment.

6. **Form of Normal Modes.** The solution of an nth degree algebraic equation offers much difficulty when n is large, especially when some of the roots are complex. Certain vector methods of solution have been developed which are helpful. When the resistances in a circuit are small, approximate methods can be employed.[1]

The roots obtained are either real or in conjugate complex pairs. If real, they are negative. A positive root would denote a current increasing without limit. This is not possible in a dissipative network. Sometimes in dealing with an infinitive distributed circuit, spurious positive roots may appear. These are considered in Chapter XIII. Complex roots will ordinarily have a negative real portion. A pair of conjugate roots may appear thus:

$$\left.\begin{aligned}\lambda_1 &= -\alpha_1 + j\omega_1 \\ \lambda_2 &= -\alpha_1 - j\omega_1.\end{aligned}\right\} \tag{54}$$

The corresponding terms of the complementary function are then

$$K_{k1}\epsilon^{(-\alpha_1+j\omega_1)t} + K_{k2}\epsilon^{(-\alpha_1-j\omega_1)t} \tag{55}$$

or

$$\epsilon^{-\alpha_1 t}(K_{k1}\epsilon^{j\omega_1 t} + K_{k2}\epsilon^{-j\omega_1 t}) \tag{56}$$

or

$$\epsilon^{-\alpha_1 t}[(K_{k1} + K_{k2})\cos\omega_1 t + j(K_{k1} - K_{k2})\sin\omega_1 t]. \tag{57}$$

In any specific solution the constants of integration, K_{k1} and K_{k2}, are also bound to come out conjugate complex quantities, so that the expression may be reduced to

$$B\epsilon^{-\alpha_1 t}\sin(\omega_1 t + \phi) \tag{58}$$

where B and ϕ are real constants of integration. This must occur, for otherwise we should have a complex expression for a real current — a result which is inadmissible in the absence of a convention by which the imaginary terms

[1] P. 94.

may be interpreted into corresponding real terms. Such a convention is often used in the symbolic solution of networks in the steady state, but it is not present here in this transient study.

The quantity α_1 is called a **natural decrement** of the network, and ω_1 is called a **natural angular velocity** of the network. In an oscillatory network there will in general be

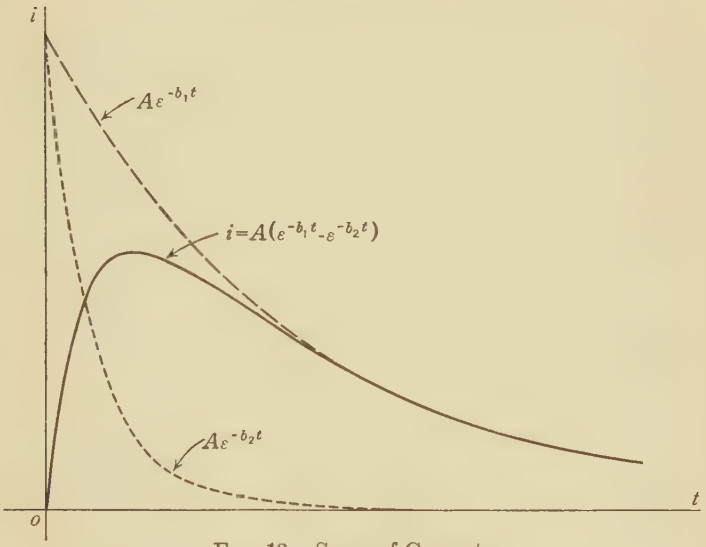

FIG. 13. Surge of Current.

as many natural angular velocities as there are circuits in the network. $\omega_1/2\pi$ is a **natural frequency** of the network. To each natural frequency corresponds a particular **damping factor** $\epsilon^{-\alpha t}$.

The combination of two complex roots gives a **damped sinusoid,** and the transient behavior of an oscillatory network is given as a sum of damped sinusoids — in general, as many as there are circuits. If there are real roots they can be similarly combined in pairs, giving a **surge** as in Fig. 13. Single, unpaired, real roots may also appear.

7. **The Amplitudes of Oscillation.** The indicial admittance of a network has now been completely determined except for the **amplitudes** K of the transient terms. The unit voltage was applied in Circuit h, and the transfer indicial admittance between Circuits h and k was desired. It has the form:

$$A_{hk}(t) = \frac{1}{Z_{hk}(0)} + K_{k1}\epsilon^{\lambda_1 t} + K_{k2}\epsilon^{\lambda_2 t} + \cdots (2n \text{ terms}). \quad (59)$$

So far the classic theory has proceeded smoothly; but when an attempt is made to determine the values of K, the amplitudes of oscillation, the classic method encounters insuperable difficulties except in very simple cases. The primary reason for the relative success of operational methods of circuit analysis lies in the fact that it avoids, to a certain extent at least, the difficulty of determining these constants of integration.

Since there are $2n$ values of K, we need $2n$ equations to determine them. One of these equations is readily written, and it will pay us to write it even though we cannot proceed to set up the others for the general case.

To find the initial conditions in the network — that is, the distribution of current at the instant just after the switch is closed — we can simplify the network by removing all condensers by short-circuiting them, and all inductances by cutting them out of circuit. This is exactly the reverse process of that for finding the final steady-state current. It follows because a condenser can support no voltage until a finite time has elapsed during which it can accumulate a charge, and an inductance can carry no finite current until the current has had a finite time to build up. It will be remembered that all condensers and inductances are considered to be initially without charge or current. The same effect of simplifying the circuit will be obtained if its response to an infinite frequency is considered, for condensers will then support no voltage and inductances will carry no

current. The initial current in the circuit will then be given by

$$\frac{1}{Z_{hk}(\infty)}.$$

From this, inserting $t = 0$ in our expression, we can obtain one equation for the K's:

$$K_{k1} + K_{k2} + \cdots = \frac{1}{Z_{hk}(\infty)} - \frac{1}{Z_{hk}(0)}. \tag{60}$$

The right-hand side of this equation is the difference between the initial and final currents. In a similar manner an equation can be written for the K's of any other branch, but they are different K's of course. By writing the relation between the current in one branch and that in another it is possible to connect them with the ones we wish to determine. This process is altogether too complex to be carried through in general. We shall content ourselves with a simple illustration.

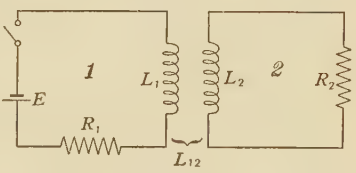

FIG. 14. Transformer Network.

8. **An Example.** Consider the transformer of Fig. 14. We seek the indicial admittance of the network from a point in one circuit. The differential equations are now

$$\left.\begin{aligned}(R_1 + L_1 p)i_1 + L_{12}pi_2 &= E.\\ L_{21}pi_1 + (R_2 + L_2 p)i_2 &= 0.\end{aligned}\right\} \tag{61}$$

The determinantal equation is

$$\begin{vmatrix} R_1 + L_1\lambda & L_{12}\lambda \\ L_{21}\lambda & R_2 + L_2\lambda \end{vmatrix} = 0 \tag{62}$$

or, since $L_{21} = L_{12}$,

$$(R_1 + L_1\lambda)(R_2 + L_2\lambda) - L^2_{12}\lambda^2 = 0 \tag{63}$$

from which can be obtained two values of λ:

$$\lambda_1, \lambda_2 = -\frac{R_1L_2 + R_2L_1}{2(L_1L_2 - L^2_{12})} \pm \frac{1}{2}\sqrt{\frac{(R_1L_2 + R_2L_1)^2}{(L_1L_2 - L^2_{12})^2} - \frac{4\,R_1R_2}{L_1L_2 - L^2_{12}}}. \tag{64}$$

Notice that L_{12}, the mutual inductance, is less than, or at most equal to, the geometric mean of the self-inductances $\sqrt{L_1 L_2}$. Thus λ_1 and λ_2 are always negative.

The impedance $Z_{11}(p)$ is

$$Z_{11}(p) = \frac{D(p)}{M_{11}(p)} = \frac{(R_1 + L_1 p)(R_2 + L_2 p) - L_{12}^2 p^2}{R_2 + L_2 p}$$

$$= R_1 + L_1 p - \frac{L_{12}^2 p^2}{R_2 + L_2 p}. \tag{65}$$

This corresponds to the well-known expression for the alternating-current impedance of a transformer:

$$Z_{11}(j\omega) = z_1(j\omega) + \frac{L_{12}^2 \omega^2}{z_2(j\omega)} \tag{66}$$

where z_1 and z_2 are respectively the impedances of primary and secondary alone. Now, inserting $p = 0$,

$$\frac{1}{Z_{11}(0)} = \frac{1}{R_1} \tag{67}$$

for the final steady state.

Also, inserting $p = \infty$, we have

$$\frac{1}{Z_{11}(\infty)} = 0 \tag{68}$$

for the initial current. In this simple example, these values are of course apparent from inspection. The indicial admittance is then

$$A_{11}(t) = \frac{1}{R_1} + K_{11}\epsilon^{\lambda_1 t} + K_{12}\epsilon^{\lambda_2 t} \tag{69}$$

and we have one equation to determine the K's:

$$K_{11} + K_{12} = -\frac{1}{R_1}. \tag{70}$$

Now the current in the secondary is

$$A_{12}(t) = K_{21}\epsilon^{\lambda_1 t} + K_{22}\epsilon^{\lambda_2 t}. \tag{71}$$

But an expression for this current can also be obtained by noting that, for each exponential term of the primary such as $K_{11}\epsilon^{\lambda_1 t}$, there is an induced secondary voltage $L_{12}\lambda_1 K_{11}\epsilon^{\lambda_1 t}$, and a term of secondary current:

$$\frac{L_{21}\lambda_1 K_{11}}{R_2 + L_2\lambda_1}\epsilon^{\lambda_1 t}.$$

Thus the secondary current can be written as

$$A_{12}(t) = \frac{L_{21}\lambda_1 K_{11}}{R_2 + L_2\lambda_1}\epsilon^{\lambda_1 t} + \frac{L_{21}\lambda_2 K_{12}}{R_2 + L_2\lambda_2}\epsilon^{\lambda_2 t}. \tag{72}$$

Now, setting this current equal to zero for the initial instant, $t = 0$, we have

$$\frac{\lambda_1}{R_2 + L_2\lambda_1}K_{11} + \frac{\lambda_2}{R_2 + L_2\lambda_2}K_{12} = 0. \tag{73}$$

There are now two equations from which K_{11} and K_{12} can be determined:

$$\left.\begin{aligned} K_{11} &= \frac{1}{R_1\dfrac{\lambda_1}{\lambda_2}\dfrac{R_2 + L_2\lambda_2}{R_2 + L_2\lambda_1} - 1} \\ K_{12} &= \frac{1}{R_1\dfrac{\lambda_2}{\lambda_1}\dfrac{R_2 + L_2\lambda_1}{R_2 + L_2\lambda_2} - 1} \end{aligned}\right\} \tag{74}$$

and the expression for the indicial admittance is completely determined.

9. **Various Initial Conditions.** Thus the classic method for the determination of the amplitudes of the modes of free oscillation drives us to special maneuvers in particular problems and gives no general method of procedure. There are many of these devices. We cannot go into them in detail here, but a few can be mentioned.

We have already seen how the network can be simplified to obtain initial and final currents. We have also seen how a current in one circuit can be transformed to find the current in another circuit by computing voltages and currents

for each exponential term that appears. There are also several other ways in which equations can be arrived at for determining the amplitudes, K. For example, the initial rate of increase of the current in a network is given by a consideration of the inductances alone. At the first instant, resistances which are in series with inductance support no voltage, for they have no current through them. The same is true for uncharged condensers which are in series with other parameters. The entire voltage is hence brought to bear on the inductances, and the rate of increase of current is given by the applied voltage divided by an equivalent inductance of the network. This equivalent inductance is obtained by combining inductances, ignoring everything else; or by setting all R's equal to zero, and all C's equal to infinity in the expression for the impedance. Thus in the example above, the impedance due to inductance alone is

$$L_1 p - \frac{L_{12}^2}{L_2} p$$

or the equivalent inductance is

$$L_1 - \frac{L_{12}^2}{L_2}.$$

The initial rate of current increase is thus:

$$\frac{1}{L_1 - \frac{L_{12}^2}{L_2}} = \lambda_1 K_{11} + \lambda_2 K_{12} \tag{75}$$

obtained by taking the derivative of the current expression and then setting $t = 0$. This equation might have been used to determine the K's in place of the one actually employed, and would have given, of course, the same results.

Other initial conditions are often useful. The initial rate of change of voltage across a condenser is the initial current divided by the capacitance. The second derivative of current in a circuit, such as the above, is given by the rate of change of the voltage applied to the inductance. This is

best illustrated in a simple series circuit as in Fig. 15. Here, when $t = 0$,

$$\left.\begin{aligned} i &= 0 & e_L &= E \\ pi &= \frac{E}{L} & pe_L &= -\frac{RE}{L} \\ p^2 i &= -\frac{RE}{L^2} & e_R &= 0 \\ e_c &= 0 & pe_R &= \frac{RE}{L} \\ pe_c &= 0 & & \\ p^2 e_c &= \frac{E}{LC} & & \end{aligned}\right\} \tag{76}$$

and so on. There are hence always plenty of ways of getting at equations for the K's in any specific problem. Unfortunately the procedure gets very complicated when there are several circuits in the network. In succeeding chapters we shall discuss general methods of arriving at the indicial admittance of networks, which will usually be found to work with less complication and difficulty. This is the main problem of the operational calculus — the ready determination of the indicial admittance.

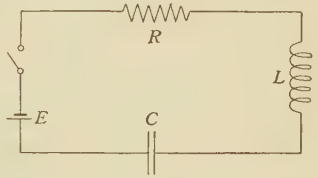

FIG. 15. Simple Series Circuit.

We have discussed in this section only networks with lumped parameters. The classic treatment of the indicial admittance of circuits with distributed parameters is so involved that we shall not review it. It is in this analysis of distributed networks that operational processes become essential if we are to proceed in a practical way.

In all the above the network has been assumed initially in equilibrium; that is, with no current in the coils or voltage across the condensers. In other words it has been assumed that there is no initial storage of energy, either in electro-

magnetic or in electrostatic form. We shall discuss the treatment of networks with initial energy storage in Chapter XIV.

10. **Impulsive Response.** Before leaving the differential equations of classic treatment there is one point to be mentioned — that of **impulsive response.** We assumed in the above treatment that all three parameters were present in each circuit. In cases where some of them are missing there is no change of treatment, except for one instance which needs comment. This is the case when $Z(\infty)$ comes out zero. The initial current is then infinite; and our treatment, in which we have assumed finite continuous expressions for the current, breaks down.

In order to meet this case, we must have across the points of voltage application a condenser unprotected by series inductance or resistance. Of course such an arrangement is a mathematical fiction, for a condenser cannot be built entirely devoid of inductance and resistance. The simple engineering way out of the difficulty is to insert in the expressions the series resistance of the condenser which is physically known to be present; and, no matter how small this resistance may be, the mathematical difficulties with infinite currents promptly disappear.

Another procedure that avoids trouble with infinite values is to solve for charges instead of currents, and obtain the current later from the derivative of charge, due regard being paid to discontinuities. When we suddenly apply unit e.m.f. to a condenser of unit capacitance, the current is mathematically infinite at $t = 0$ and zero thereafter. We shall find that this fundamental sort of impulsive response corresponds to a simple operational expression, the presence of which has sometimes caused perplexity. Yet if we reason this proposition in terms of charges, much of the mystery vanishes. At $t = 0$ when the switch is closed, a unit charge is (mathematically) instantly placed on the condenser, and that is all.

This case of impulsive response may occur in all the theorems which we shall derive on the basis of the differential equation formulation. It has been customary to make an exception to each rule or formula in order to bar such cases. We shall find, however, that by adding a special term, it is usually possible to include impulsive response in the regular treatment and avoid continual exceptions.

PROBLEMS ON CHAPTER IV

Prob. 1–4. For the circuit of Fig. 1, what is the determinantal equation? What are the normal modes, the natural decrements, the natural frequencies, and the damping factors?

Prob. 2–4. Sketch the following for positive and negative values of t: $t\,1$, $\epsilon^{-t}\,1$, $5\,t^3\,1$, $t^{-2}\,1$, and $(1 - \epsilon^t)\,1 + \epsilon^t$.

Prob. 3–4. How many normal modes has the circuit of Fig. 4? What initial conditions might be used in determining K_1 and K_2?

Prob. 4–4. Obtain i_1 for Fig. 2, using the following values: $L_1 = 0$, $L_2 = 30$ h., $L_{12} = 10$ h., $R_1 = 10$ ohms, $R_2 = 15$ ohms, and $E = 230$ volts. Close the switch at $t = 0$.

Prob. 5–4. Determine i_1 and i_2 in Fig. 1, using these values: $L_1 = L_2 = 0$, $R_1 = 1$ ohm, $R_2 = 30$ ohms, $R_{12} = 10$ ohms, $C_1 = 50$ mfd., $C_2 = 10$ mfd., and $E = 6$ volts. The switch is closed at $t = 0$. What is the indicial admittance of the network viewed from the battery?

Prob. 6–4. Find $A_{11}(t)$ and $A_{12}(t)$ for the coupled circuit of Fig. 14, adding a condenser of 5×10^{-10} farads in series with the secondary. The parameters are $R_1 = 10$ ohms, $R_2 = 5$ ohms, $L_1 = 0.5$ h., $L_2 = 1.0$ h., $L_{12} = 0.5$ h. Take $E = 0.1$ volt. The switch is closed at $t = 0$.

Prob. 7–4. Find the velocity of the 300-gram pendulum bob, in Prob. 8–2 as a function of time.

CHAPTER V

THE SUPERPOSITION THEOREM

We have seen how the **principle of superposition** may be used in the analysis of networks with **fixed parameters.**[1] According to this principle, the total effect of several simultaneously-applied causes may be obtained by adding the effects of each acting independently. In electrical terms, if several sources of e.m.f. act in the same **linear network** at the same time, they will produce a current which will be the sum of the currents produced by each e.m.f. acting alone.

There is an important theorem which holds for any network of this sort, and which enables us to find the response of the network to an applied electromotive force which varies in any manner whatever, provided we know the response of the network to unit e.m.f.; that is, if we know its indicial admittance $A(t)$. This theorem may be called the **superposition theorem.** It was early employed by Boltzmann[2] and Duhamel,[3] and is much used in mathematics. Hopkinson[4] applied it to electric problems.

If $A(t)$ expresses the current which flows in a network due to a unit e.m.f. suddenly impressed, then the current $i(t)$ that will flow when the impressed voltage is given by the time function $e(t)$ will be

$$i(t) = e(0)A(t) + \int_0^t A(t - \lambda)\, e'(\lambda)\, d\lambda. \qquad (77)$$

[1] Chapter II.

[2] L. Boltzmann, Zur Theorie der elastischen Nachwirkung, Wien. Ber. **70,** 1874, p. 275; Wissenschaftliche Abhandlungen, Barth, Leipzig, 1909, I, p. 621.

[3] J. Éc. Polytech., Paris, **14,** Cah. 22, 1833, p. 20.

[4] J. Hopkinson, Residual Charge of a Leyden Jar, Phil. Trans., London, **167,** 1877, p. 599; or Original Papers, Cambridge University Press, 1901, II, p. 13.

This equation may be written in several forms, of which this is one of the most convenient. Others will be developed below.

In equation (77), $e'(\lambda)$ means the derivative of the function $e(\lambda)$ with respect to λ — that is, the slope of the electromotive force curve. $e(0)$ is the value of the e.m.f. at $t = 0$, and will of course be zero if the curve for $e(t)$ passes through the origin. $A(t)$ in the expression can be any indicial admittance of the network. It may be a transfer indicial admittance, making possible the determination of the current at a distant point of the network due to the local application of the e.m.f. $e(t)$. As many engineers seldom employ integrals with a variable parameter under the sign, some effort will be devoted to making the meaning of this equation clear.

In the first place, λ in this expression is a variable of integration only. It disappears from the expression when the integration is performed, and hence may be replaced by any other symbol without changing the meaning of the equation in the least. For example,

$$i(t) = e(0)\, A(t) + \int_0^t A(t - x)\, e'(x)\, dx. \tag{78}$$

In an ordinary definite integral of the product of two functions as, for example,

$$\int_0^t f_1(\lambda)\, f_2(\lambda)\, d\lambda \tag{79}$$

we prescribe that the following operation be performed graphically or formally. In graphical evaluation, the two curves representing $f_1(\lambda)$ and $f_2(\lambda)$ are to be multiplied together point by point to obtain a product curve. The area under this product curve is to be measured between the ordinates erected at 0 and t, and the result is the value of the integral for that value of t. For a new value of t, a new area up to a new ordinate must be found, employing the same product curve as before.

When there is a variable parameter under the sign of integration, the graphical interpretation differs from this procedure. Suppose $e'(\lambda)$ is represented by the curve of Fig. 16*a* and $A(\lambda)$ by the curve of Fig. 16*b*. Then for a given value of the variable parameter t, the plotted curve of Fig. 16*c* represents $A(t-\lambda)$. The curves of *a* and *c* must now be multiplied together, giving a curve such as Fig. 16*d*. Then the area under this curve between ordinates at 0 and t, as shown cross-hatched, must be obtained. This gives the value of the integral for the given value of t.

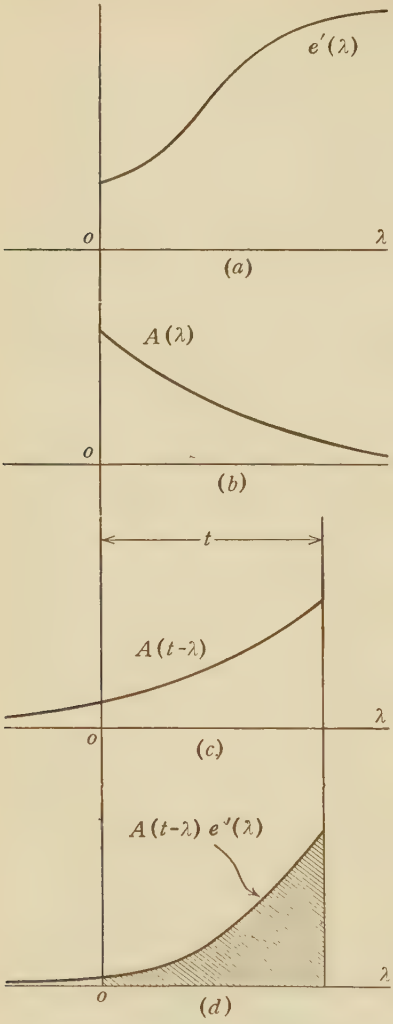

FIG. 16. Plots of Integrand of Equation (77).

But now for a new value of t, we must start over, plot a new curve in place of Fig. 16*c*, which will of course be simply shifted in place, plot an entirely new product curve in place of Fig. 16*d*, and take the new area. Finally these results for various value of t are plotted against t to give the curve which represents the integral.

This fact — that a new curve bounds the area defined by the integral for each value of t — makes it relatively difficult to evaluate by graphical or mechanical processes any integral with a variable parameter under the sign.

Of course the usual method is to evaluate the integral formally wherever this is possible. Let us apply the theorem to a very simple case. Consider a condenser C in series with a resistance R, and let us apply to this combination an alternating e.m.f. $E \sin \omega t$, the switch being closed at the zero of voltage, so that $e(0) = 0$.

$A(t)$ for this circuit is a well-known result:

$$A(t) = \frac{1}{R} \epsilon^{-\frac{t}{RC}}. \tag{80}$$

Also,

$$e'(\lambda) = \omega E \cos \omega \lambda \tag{81}$$

which can be written

$$e'(\lambda) = \frac{\omega E}{2} (\epsilon^{j\omega\lambda} + \epsilon^{-j\omega\lambda}). \tag{82}$$

The superposition theorem then gives

$$\begin{aligned} i(t) &= \frac{\omega E}{2R} \int_0^t \epsilon^{-\frac{t-\lambda}{RC}} (\epsilon^{j\omega\lambda} + \epsilon^{-j\omega\lambda}) \, d\lambda \\ &= \frac{\omega E}{2R} \epsilon^{-\frac{t}{RC}} \int_0^t \left(\epsilon^{\frac{\lambda}{RC} + j\omega\lambda} + \epsilon^{\frac{\lambda}{RC} - j\omega\lambda} \right) d\lambda. \end{aligned} \tag{83}$$

Performing the integration, which now involves no variable parameter since the factor containing t has been properly moved outside the sign of integration:

$$\begin{aligned} i(t) = \frac{\omega EC}{2} &\left(\frac{1}{1 + j\omega RC} \epsilon^{j\omega t} + \frac{1}{1 - j\omega RC} \epsilon^{-j\omega t} \right) \\ &- \frac{EC\omega}{1 + R^2C^2\omega^2} \epsilon^{-\frac{t}{RC}} \end{aligned} \tag{84}$$

which can be reduced to

$$i(t) = \frac{EC\omega}{1 + R^2C^2\omega^2}\left[(\cos \omega t + RC\omega \sin \omega t) - \epsilon^{-\frac{t}{RC}}\right]$$

or

$$i(t) = E\left\{\frac{1}{\sqrt{R^2+\frac{1}{C^2\omega^2}}}\sin(\omega t+\phi) - \frac{1}{\omega C\left(R^2+\frac{1}{C^2\omega^2}\right)}\epsilon^{-\frac{t}{RC}}\right\} \tag{85}$$

where $$\phi = \tan^{-1}\frac{1}{RC\omega}$$

which is the usual expression for the alternating-current transient of a condenser circuit. Note that the superposition theorem gives the entire expression, steady-state term and transient.

2. **Derivation of Superposition Theorem.** The derivation of the superposition theorem is not difficult. First let us consider some simple preliminary matters.

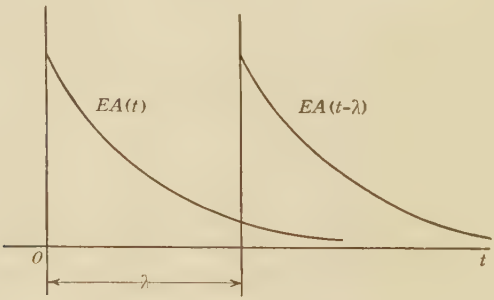

FIG. 17. Shift of Axis.

Let the indicial admittance of a network be $A(t)$, and suddenly apply a constant e.m.f. E at $t = 0$. This produces a current $EA(t)$. An e.m.f. of magnitude E, but impressed at a later time, say when t has the value λ, will evidently produce a current $EA(t)$, but shifted λ units to the right as in Fig. 17.

We are now to consider the application to this network of an e.m.f. which varies in accordance with a known time function $e(t)$. This can be considered in two parts, the first being the sudden jump in voltage when the switch is closed, or $e(0)$. This produces a current term $e(0)A(t)$. This first part is zero if there is no initial jump in voltage; that is, if $e(t)$ passes through the origin. The effect of the remainder can be added to this by the principle of superposition to give the total result.

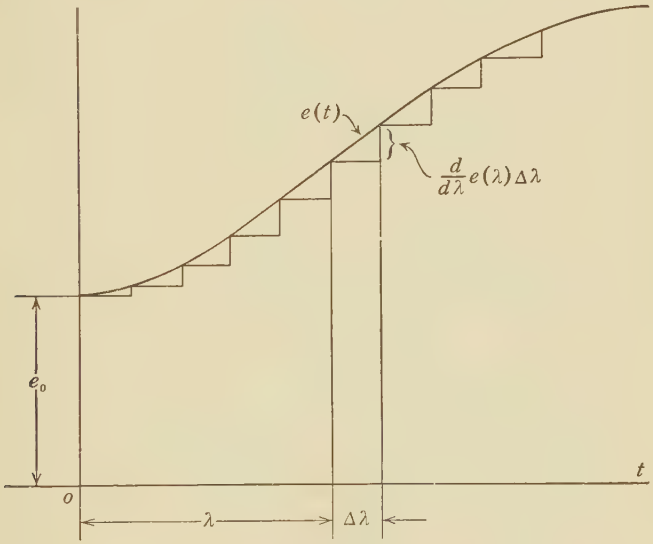

FIG. 18. Step Function Approximation.

An approximate expression can be obtained by finding the effect of a step function which closely fits the given function, as in Fig. 18, so that the total voltage is treated as a sum of a large jump at $t = 0$, plus a lot of similar but smaller jumps occurring at later times. Divide the horizontal axis into short lengths and consider one of these steps at a distance λ from the origin. Notice that the symbol t has already been used to denote the value of the time at which we are attempt-

ing to find the value of the current. The point on the curve to which attention is now directed corresponds to a previous time, so it cannot be designated by the same symbol. We therefore use λ. The horizontal dimension of the step at this point is thus called $\Delta\lambda$. The vertical height of the step will then be obtained by multiplying this horizontal distance by the slope of the curve at this point, or

$$\frac{d}{d\lambda} e(\lambda)\, \Delta\lambda$$

or

$$e'(\lambda)\, \Delta\lambda.$$

This particular jump in voltage gives a current term, which at the time t will have the value:

$$A(t - \lambda)\, e'(\lambda)\, \Delta\lambda.$$

To get the total effect at the time t, we need to add the effect of all the jumps up to this time; so our approximate expression for the current becomes

$$i(t) = e(0)\, A(t) + \sum_{0}^{t} A(t - \lambda)\, e'(\lambda)\, \Delta\lambda. \qquad (86)$$

Now take the size of the steps smaller and smaller. The step function approaches the value of our known function of e.m.f., and the above expression approaches the integral expression:

$$i(t) = e(0)\, A(t) + \int_{0}^{t} A(t - \lambda)\, e'(\lambda)\, d\lambda \qquad (77)$$

which is the superposition theorem.

For mathematical rigor it might be well to consider also the step function formed on the other side of the curve; that is, the step function formed with the jagged line above the smooth curve, and to study with care the statement that the limit of the effect produced by a step function is actually the result produced by the smooth function. A physical

discussion, however, will treat this point better for our present purposes, and will point out the one case where we must be on our guard.

3. **One Case Requiring Special Treatment.** Suppose we are asked to perform the experiment of suddenly applying to a network an exponentially-varying voltage. We would probably proceed as follows: We would arrange a potentiometer capable of giving voltages throughout the range specified for the experiment, connect it to the network, and move the handle exponentially; as, for example, by using a properly-cut cam. Suppose we utilize for this purpose an ordinary resistance tube wound with wire with a slider resting on the turns of wire. We shall actually apply to the network, not a smooth variation of voltage, but exactly the sort of step-by-step variation we have discussed. Yet, if the wire of the rheostat is small enough so that the steps are very small, we should not hesitate usually to consider the effect very closely approximate to that which would be produced by a smooth variation, and we should accept the process of passing to the limit as sound.

We should hesitate in just one case. Suppose the network consisted of a pure capacitance with very small series resistance; or was of a more complicated nature, still having the properties of a condenser at the instant when voltage was suddenly applied. This is the network of impulsive response; which, as we noted in Chapter IV, requires special treatment.

As the slide wire moves, the effect of the steps is to produce a voltage ripple superposed on a smooth variation. This ripple produces a ripple of current, and as the amplitude of the voltage ripple becomes smaller the amplitude of the current ripple will ordinarily also decrease. But with a pure condenser this is not true. The magnitude of the alternating-current impedance of such an element is $1/C\omega$, and it is hence small for a high frequency. As we decrease the size of the steps, the impedance goes down at the same rate as

the amplitude of voltage ripple; so the amplitude of current variation does not decrease. Of course this reasoning is only approximate, for the effect of the steps is not exactly given by analysis in terms of an alternating e.m.f. However, it is certainly doubtful if we are here justified in utilizing the process of passing to the limit.

It is as evident mathematically that such a case requires special treatment. For a circuit of impulsive response, $A(t)$ has an infinite value at $t = 0$, and is finite thereafter. In the integral expression integration is specified up to such a point of discontinuity. This cannot be dealt with offhand.[1]

4. **Form for Impulsive Network.** When a network is capable of **impulsive response** a special form of the superposition theorem can be obtained as follows:

First consider a pure capacitance C, and an applied e.m.f. which will be called $e(t)$ and which will be zero when $t = 0$ — that is, $e(0) = 0$. Since

$$i(t) = C\frac{de}{dt} \tag{87}$$

the current in the condenser is

$$i(t) = Ce'(t). \tag{88}$$

Next suppose a unit e.m.f. is applied to such a condenser. We have seen in Chapter IV, Sec. 10, that the current is in the form of a pure impulse, infinite at $t = 0$, and zero thereafter. In terms of charge, the response consists of the instantaneous transfer of a charge of amount C to the plates of the condenser. The charge on the condenser plates, written as a time function, is thus equal to C multiplied by the unit function, or $C1$. Since the current is the derivative of charge, the current which flows when unit e.m.f. is applied to a pure condenser is

$$i(t) = Cp1. \tag{89}$$

[1] F. D. Murnaghan, The Boltzmann-Hopkinson Principle of Superposition as Applied to Dielectrics, Trans. A.I.E.E., **47**, 1928, p. 366.

At this point we do not need to inquire deeply into the legitimacy of writing a current expression in this way; it is simply a convenient manner of specifying an impulsive current.

Finally let us consider that an e.m.f. $e(t)$ which has a finite value at $t = 0$, so that $e(0) \neq 0$, is suddenly applied to the pure condenser. Evidently the response will be in two parts — an initial pulse proportional to C and to $e(0)$, plus a term depending on the subsequent change of voltage, or

$$i(t) = Ce(0)p1 + Ce'(t). \tag{90}$$

This is the superposition theorem for a pure condenser.

A network can have an impulsive response provided there is some path between terminals involving pure condensers only. The network may then be thought of in two parts: the condenser group which includes all the pure condensers which may be considered in this path without bringing in elements with other parameters, and the remainder of the network which may be connected into the condenser group at various points. An example of such a network is shown in Fig. 19. Here the condenser group includes elements C_1 and C_2.

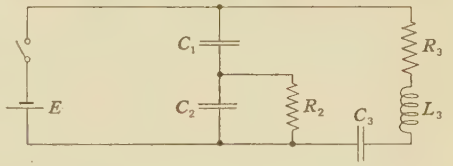

FIG. 19. Network with Condenser Group.

Consider the application of unit e.m.f. to this sort of network. Evidently there will be an initial pulse of current $\bar{C}p1$, where $\bar{C}$ is the **equivalent capacitance of the condenser group**. This pulse instantaneously places various charges on the condensers of the group, and therefore brings to bear various voltages on the remainder of the network. After $t = 0$ there will be, in general, a readjustment with a finite current in various parts of the network, and hence even at the terminals where the unit e.m.f. is applied. The evaluation of the driving-point indicial admittance of this sort of

network consists of an initial pulse plus a succeeding finite current. For the present it is assumed that we know how to obtain this indicial admittance, which we write:

$$A(t) = \overline{A}(t) + \overline{C}p1 \tag{91}$$

so that $\overline{A}(t)$ is only the finite portion of the indicial admittance, the impulse being taken care of by the second term. What is the response when $e(t)$ is applied to the same network?

Leave out of account for the moment the impulsive term, and glance back at the derivation of the superposition theorem. It will be seen that each step follows as before, provided we use the finite $\overline{A}(t)$ in the expressions. But the impulsive part is due to the pure condenser group, and thus there must be added to the previous result an expression such as has been just obtained for a pure condenser. The superposition theorem for a network of impulsive response, considering the driving-point indicial admittance, thus becomes

$$i(t) = e(0)\,\overline{A}(t) + \int_0^t \overline{A}(t-\lambda)e'(\lambda)\,d\lambda + \overline{C}e'(t) + \overline{C}e(0)\,p1. \tag{92}$$

Of course if $e(0) = 0$, the first and last terms drop out. This may be considered the general form; and for an ordinary network which is non-impulsive, $\overline{C} = 0$ and $\overline{A}(t) = A(t)$, so that the expression reverts to the previous form.

The preceding analysis has been for effects at the terminals where the e.m.f. is applied, only the driving-point indicial admittance being considered. The superposition theorem may be employed with an impulsive network in cases where a transfer indicial admittance is used. Two cases may rise. In the first, the point at which the current is to be evaluated lies in the condenser group; while in the second, it lies somewhere in the remainder of the network. In either case the reasoning previously employed can be reproduced

exactly, assuming of course that we are not concerned with methods of finding the indicial admittance but only with ways of employing it to find the response to an applied e.m.f. of general form. In case the current is examined at a point in the condenser group, all four terms of the superposition theorem will appear; but a new value of $\bar{C}$, which we may call if we please a **transfer equivalent capacitance**, must be used to account for the fact that not all of the entering current (corresponding to the last two terms) may reach the point in question. This can be evaluated by examining the way in which the current will divide when a steadily-varying voltage is employed at the terminals, considering the condenser group alone apart from the remainder of the network. Or, more simply, the subdivision of current may be considered when an alternating e.m.f. is applied to the condenser group only. The ratio of the current at the point in question to the terminal current then gives the ratio by which the total equivalent capacitance of the condenser group should be multiplied to obtain the transfer equivalent capacitance. In case the point at which the current is to be examined lies in the remainder of the network, these last two terms do not appear and this discussion is unnecessary.

Later, when the direct operational method of finding $A(t)$ is studied, we shall see that for an impulsive case both $\overline{A}(t)$ and $\overline{C}$ are automatically evaluated without any necessity for deriving them separately.

Of course all difficulty with the impulsive network can be removed by inserting in the diagram of the network the small resistance which is always actually present in series with the condensers and analyzing accordingly, when there will be no impulsive response to consider. However, it will later be convenient to consider in some cases the mathematical fiction of a pure condenser. We shall ordinarily write the superposition theorem without the extra term and consider impulsive circuits only when we so state explicitly.

5. **Transformation into Various Forms.** The **superposition theorem** can be transformed into **various forms.** We have

$$i(t) = e(0)\,A(t) + \int_0^t A(t-\lambda)\,e'(\lambda)\,d\lambda. \tag{77}$$

Let

$$\left\{\begin{array}{r} \lambda = t - x \\ d\lambda = -dx \\ t - \lambda = x. \end{array}\right.$$

In making this change, the limits must also be changed. When $\lambda = 0$, $x = t$; so the lower limit becomes t. When $\lambda = t$, $x = 0$; so the upper limit becomes 0. This involves a change of sign. So

$$i(t) = e(0)\,A(t) + \int_0^t A(x)\,e'(t-x)\,dx \tag{93}$$

which may as well be written

$$i(t) = e(0)\,A(t) + \int_0^t A(\lambda)\,e'(t-\lambda)\,d\lambda. \tag{94}$$

Integrate the second term of (77) by parts:

$$\int_0^t A(t-\lambda)\,e'(\lambda)\,d\lambda = A(t-\lambda)\,e(\lambda)\Big|_0^t - \int_0^t A'(t-\lambda)\,e(\lambda)\,d\lambda$$
$$= A(0)\,e(t) - A(t)\,e(0) - \int_0^t A'(t-\lambda)\,e(\lambda)\,d\lambda \tag{95}$$

where $A'(t-\lambda)$ means $\dfrac{\partial}{\partial\lambda}A(t-\lambda)$.

Combining, there results:

$$i(t) = e(t)\,A(0) - \int_0^t A'(t-\lambda)\,e(\lambda)\,d\lambda. \tag{96}$$

This may also be written

$$i(t) = e(t)\,A(0) + \int_0^t \frac{\partial}{\partial t}A(t-\lambda)\,e(\lambda)\,d\lambda \tag{97}$$

since

$$\frac{\partial}{\partial t} A(t - \lambda) = -\frac{\partial}{\partial \lambda} A(t - \lambda). \tag{98}$$

The same process of change of variable will alter this form to

$$i(t) = e(t)\, A(0) + \int_0^t A'(\lambda)\, e(t - \lambda)\, d\lambda. \tag{99}$$

Other forms can be obtained thus:

$$\frac{d}{dt}\int A(t - \lambda)\, e(\lambda)\, d\lambda = A(0)\, e(t) + \int_0^t \frac{\partial}{\partial t} A(t - \lambda)\, e(\lambda)\, d\lambda. \tag{100}$$

This follows from the rule for the derivative of a definite integral with variable limits and a parameter under the sign.[1]

Combining with the above:

$$i(t) = \frac{d}{dt}\int_0^t A(t - \lambda)\, e(\lambda)\, d\lambda. \tag{101}$$

In this last expression use the change of variable, and obtain

$$i(t) = \frac{d}{dt}\int_0^t A(\lambda)\, e(t - \lambda)\, d\lambda. \tag{102}$$

We now have six principal forms of the superposition theorem. Their interrelations are well studied by applying each

[1] If

$$\phi(t) = \int_{t_0}^{t_1} f(t,\lambda)\, d\lambda, \text{ then}$$

$$\frac{d\phi}{dt} = \int_{t_0}^{t_1} \frac{\partial f}{\partial t}\, d\lambda - f(t,t_0)\frac{dt_0}{dt} + f(t,t_1)\frac{dt_1}{dt}.$$

Thus, if

$$\phi(t) = \int_0^t A(t - \lambda)\, e(\lambda)\, d\lambda,$$

$$\frac{d}{dt}\int_0^t A(t - \lambda)\, e(\lambda)\, d\lambda = A(0)\, e(t) + \int_0^t \frac{\partial}{\partial t} A(t - \lambda)\, e(\lambda)\, d\lambda.$$

See, for example, E. B. Wilson, Advanced Calculus, Ginn & Co., 1912, p. 283; or F. S. Woods, Advanced Calculus, Ginn & Co., 1926, p. 142.

form graphically to a simple problem, such as that of Section 1 of this chapter.

6. **Sinusoidal Applied Electromotive Force.** When the applied electromotive force is **sinusoidal,**

$$e(t) = E \sin(\omega t + \theta)$$

there are other useful forms.

Inserting this in (102), we have

$$i(t) = E \frac{d}{dt} \int_0^t A(\lambda) \sin[\omega(t - \lambda) + \theta] \, d\lambda. \qquad (103)$$

Now

$$\sin(\omega t - \omega\lambda + \theta) = \sin(\omega t + \theta) \cos \omega\lambda - \cos(\omega t + \theta) \sin \omega\lambda$$

and on inserting this and taking out from under the integral sign parts not containing λ:

$$i(t) = E \frac{d}{dt} \Big[\sin(\omega t + \theta) \int_0^t \cos \omega\lambda A(\lambda) \, d\lambda - \cos(\omega t + \theta) \int_0^t \sin \omega\lambda A(\lambda) \, d\lambda \Big]. \qquad (104)$$

We now take the derivative, remembering to include the effect of the variable limit, and obtain

$$i(t) = E \sin\theta A(t) + E\omega \cos(\omega t + \theta) \int_0^t \cos \omega\lambda A(\lambda) \, d\lambda + E\omega \sin(\omega t + \theta) \int_0^t \sin \omega\lambda A(\lambda) \, d\lambda. \qquad (105)$$

Other forms for the sinusoidal case can also be readily obtained.

If the form of $A(t)$ is such that the infinite integrals converge, this expression can be written

$$i(t) = E \sin\theta A(t) - E\omega \cos(\omega t + \theta) \int_t^\infty \cos \omega\lambda A(\lambda) \, d\lambda - E\omega \sin(\omega t + \theta) \int_t^\infty \sin \omega\lambda A(\lambda) \, d\lambda$$

$$+ E\omega \cos(\omega t + \theta) \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda$$

$$+ E\omega \sin(\omega t + \theta) \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda. \qquad (106)$$

The last two terms alone contain t only in the sine and cosine expressions. These then give the **steady-state current,** and the other three terms give the **transient current.** If $i_s(t)$ denotes the steady-state current and $i_t(t)$, the transient,

$$i_s(t) = E\omega \cos(\omega t + \theta) \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda$$

$$+ E\omega \sin(\omega t + \theta) \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda \qquad (107)$$

and

$$i_t(t) = E \sin \theta A(t)$$

$$- E\omega \cos(\omega t + \theta) \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda$$

$$- E\omega \sin(\omega t + \theta) \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda. \qquad (108)$$

In these forms of the superposition theorem for sinusoidal applied e.m.f., the integrals involve no variable parameters under the sign of integration, and hence they may be evaluated in the case of any numerical problem by the usual mechanical or graphical processes for evaluating definite integrals.[1]

If $A(t)$ contains a constant term, the procedure can still be handled as will be outlined in Chapter X, although these present expressions may contain integrals which cannot be evaluated. Of course these forms can be used only when $A(\lambda)$ is such that the infinite integrals converge. If $A(\lambda)$ has a constant term, it may be split off and the corresponding terms evaluated separately in order to avoid this difficulty.

[1] See, for instance, J. Lipka, Graphical and Mechanical Computation, John Wiley, 1918, Chapter IX; F. S. Dellenbaugh, Jr., Harmonic Analysis, M.I.T., S. M. Thesis, 1921.

7. **Simplification for Lumped Network.** For any network with lumped fixed parameters the form of $A(t)$ is

$$A(t) = \frac{1}{Z(0)} + K_1\epsilon^{\lambda_1 t} + K_2\epsilon^{\lambda_2 t} + \cdots \tag{109}$$

where $Z(0),\ K_1, K_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$

are all constants. The λ's are found from the determinantal equation, and may be real or complex.

The form of the superposition theorem can be altered by inserting this expression for $A(t)$. Using the form:

$$i(t) = e(0)A(t) + \int_0^t A(t-\lambda)\, e'(\lambda)\, d\lambda \tag{77}$$

we have

$$i(t) = e(0)A(t) + \frac{1}{Z(0)}\int_0^t e'(\lambda)\, d\lambda + \int_0^t [K_1\epsilon^{\lambda_1(t-\lambda)} + K_2\epsilon^{\lambda_2(t-\lambda)} + \cdots]\, e'(\lambda)\, d\lambda. \tag{110}$$

Again parts not containing λ may be taken from under the integral sign. So

$$i(t) = e(0)A(t) + \frac{1}{Z(0)}[e(t) - e(0)] + K_1\epsilon^{\lambda_1 t}\int_0^t \epsilon^{-\lambda_1\lambda}\, e'(\lambda)\, d\lambda + K_2\epsilon^{\lambda_2 t}\int_0^t \epsilon^{-\lambda_2\lambda}\, e'(\lambda)\, d\lambda + \cdots. \tag{111}$$

Again the integrals have been reduced to the form of ordinary definite integrals. A similar procedure can be carried out when $A(t)$ is expressed in the form where terms involving complex roots are grouped into damped sinusoids.

The superposition theorem is not limited in application, however, to circuits with lumped parameters; it applies even when parameters are distributed, provided only that they are fixed. It is not in general applicable, of course, to circuits with variable parameters.[1]

[1] A special form of the theorem can be derived for use when there is a parameter present variable with the time.

8. **Applied Currents.** A superposition theorem can be written in which current and voltage are reversed in position, and in which the indicial admittance is replaced by an indicial impedance, as will be shown below. In the form previously utilized, involving $A(t)$, $e(t)$ is known and the solution is for $i(t)$. It is entirely possible, however, that the problem of finding $e(t)$ may be encountered when $i(t)$ is known and when the properties of the network are specified by means of $A(t)$. The above equation (77) may be used for such a problem, but the unknown now appears under the sign of integration. This is an integral equation, other examples of which are met in Chapter XV where schemes of solution are treated briefly.

PROBLEMS ON CHAPTER V

Prob. 1–5. By use of the superposition theorem find the current due to an e.m.f. of $E \cos \omega t$ suddenly applied at $t = 0$ to a series circuit of R and L.

Repeat using an impressed e.m.f. $E\epsilon^{-\alpha t} \sin \omega t$ where α is a positive real.

Prob. 2–5. For the example of Sec. 1, Chapter V, sketch the $A(t)$, $e(t)$, and $i(t)$ functions for all six forms of the superposition theorem.

Prob. 3–5. A circuit consists of a pure condenser C shunted by a resistance R. An e.m.f., $E \cos \omega t$, is suddenly applied to the condenser terminals at $t = 0$. What is the current through the source?

Prob. 4–5. Consider a network consisting of two condensers C_1 and C_2 in series, and across the latter of which is connected a resistance R in series with ar inductance L. The e.m.f. applied to the two condensers in series is

$$e = 200 \sin 377\, t.$$

The constants are $C_1 = C_2 = 10$ mfd., $L = 5$ h., and $R = 1000$ ohms. Find the generator current when the e.m.f. is applied at $t = 0$. Find the current through the resistance for this case.

Prob. 5–5. An R, L, and C series circuit which is oscillatory is suddenly subjected to a sinusoidal e.m.f. having a frequency which coincides with the natural frequency of the circuit, the e.m.f. being applied at the maximum point of the voltage wave. By means of the superposition theorem set up a general expression for the response.

Prob. 6–5. The loud-speaker apparatus of Prob. 9–3 is used to reproduce a piano solo. Plot the velocity of the loud-speaker diaphragm due to a single note which produces the voltage

$$e = E\,[\epsilon^{-10\,t} \sin 600\,t + \frac{1}{3}\,\epsilon^{-100\,t} \sin 1200\,t].$$

What can you say as to the distortion?

Prob. 7–5. Determine $i_s(t)$ and $i_t(t)$ from equations (107) and (108), Chapter V, for the R and L series circuit. Consider the voltage impressed to be $100 \sin (377\,t + 30°)$, and that $R = 10$ ohms and that $L = 0.1$ h.

Prob. 8–5. Heaviside, Electromagnetic Theory, Vol. II, p. 54, treats a problem in which a peculiar form of voltage impressed on a cable results in an initial pulse of current, and no current thereafter. A form of voltage can be found to accomplish the same thing with various lumped networks; for example, a pure condenser shunted by a resistance, or with somewhat more complication, shunted by a resistance and inductance in series. Examine this matter for various simple networks.

Prob. 9–5. The oscillograph of Prob. 15–2, is tested by suddenly impressing a constant current and recording an oscillogram. The oscillograph is now to be used to record a sinusoidal current which is suddenly impressed. How can the error in the record be predetermined from the results of the test, assuming the exact value of the sinusoid is known? If you have the converse problem — a record of an unknown current variation to find the exact way in which the current varied — how could you proceed to obtain an approximate correction, assuming the error to be small compared to the amplitude of the record?

CHAPTER VI

THE INFINITE INTEGRAL THEOREM

Since either the impedance function Z or the indicial admittance A completely determines the performance of any linear network, it follows that there must be a definite relationship between them. There are two very important relationships of this sort, one of which will be treated in this chapter. It expresses the relation by means of an infinite integral, and it will hence be called the **infinite integral theorem,** or the **integral theorem** for short.

This theorem first appeared in Heaviside's Electromagnetic Theory.[1] He there derived it by operational processes which we have not yet treated, and we shall therefore reserve comment on his method until Chapter VIII. The theorem follows immediately from the consideration of networks in terms of the Fourier Integral.[2] Carson[3] has devoted much attention to this special theorem, and in fact makes it the central feature of the operational method. In this present chapter we shall consider the general meaning and scope of the theorem, and derive it directly from the superposition theorem by a method very similar to that used by Carson.

The impedance function of a network can be obtained very easily in nearly every case. In the preceding chapter we saw that when A is known the response of the network to any applied electromotive force can be found. If, then, A

[1] Vol. III, p. 237.

[2] We shall pay attention to this aspect in Chapter X.

[3] J. R. Carson, Theory of Transient Oscillations of Electrical Networks, Trans. A.I.E.E., **38,** 1919, p. 345; The Heaviside Operational Calculus, B.S.T.J., **1,** 1922, #2, p.43; The Heaviside Operation Calculus, Bull. Am. Math. Soc., **31,** 1926, p. 43. See also Electric Circuit Theory and Operational Calculus, McGraw-Hill, 1926. This excellent book is especially valuable to students with good mathematical preparation who will find it clear and delightful.

can be determined from Z, the procedure will be complete. This step, however, does not prove to be easy, and nearly all of the Heaviside Operational Calculus is devoted to methods of taking this step conveniently.

There are many ways in which Heaviside took this step. He did not use the integral theorem extensively for this purpose, probably because he arrived at its formulation rather late in his work, but also undoubtedly because he preferred the other methods. Therefore he used this theorem as a powerful scheme for evaluating infinite integrals when the operational equation was known, rather than as a method for establishing the operational processes themselves. Evidently it can be used in either direction, and we shall find it most useful here as a secure basis on which to place operational processes that might otherwise be of doubtful validity. We therefore consider it now in order that it may be available as we study other schemes, especially the expansion theorem and the direct operational method, which are the actual working tools of the operational calculus.

The direct operational methods will be found to be most powerful. They were introduced by Heaviside in a remarkable series of articles and in his Electromagnetic Theory, which constitute the classics of circuit theory. Heaviside wanted results and he certainly obtained them. A generation before the real appreciation of such matters, he examined mathematically the behavior of power and communication circuits and worked out subjects as involved as the loading of lines and the progress of traveling waves and surges. He was an accomplished mathematician, but he did not hesitate to use methods which were on the borderland of known processes; and he introduced mathematical procedure of a radical and powerful sort. Some of his methods have not been completely ironed out even to-day, and many of them were substantiated only recently, principally through the activities of such workers as Carson, Wiener, Wagner, Fry, Bromwich, Eccles, Pleijel, Giorgi, and several others.

It has not been always or everywhere that the monumental nature of Heaviside's work has been appreciated, and the efforts of men such as Behrend,[1] Vallarta, and Berg, in insisting upon a due measure of appreciation of his genius have done much to make Heaviside's work used as well as useful. The direct operational method was produced by Heaviside largely experimentally, and he sometimes found what processes would work only by trying them on known problems. He was not afraid to make mistakes, if he also made real advances. It is of course far better that operational processes should rest on a secure foundation. The integral theorem is an outstanding feature which has furnished a great part of such a foundation. We shall, therefore, treat this theorem first.

2. **Derivation from Superposition Theorem.** We saw in the last chapter that the steady-state response of a linear network to a sinusoidal impressed e.m.f. may be written

$$i_s(t) = E\omega \cos(\omega t + \theta) \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda + E\omega \sin(\omega t + \theta) \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda. \qquad (107)$$

Write this for $\theta = 0$, and $E = 1$:

$$i_s(t) = \omega \cos \omega t \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda + \omega \sin \omega t \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda. \qquad (107a)$$

Putting in the exponential form of the sine and cosine, this becomes

$$i_s(t) = \frac{\omega}{2}(\epsilon^{j\omega t} + \epsilon^{-j\omega t}) \int_0^\infty \cos \omega\lambda A(\lambda)\, d\lambda + \frac{\omega}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) \int_0^\infty \sin \omega\lambda A(\lambda)\, d\lambda. \qquad (107b)$$

[1] See especially, B. A. Behrend, Career of Oliver Heaviside, El. Wld., **85,** 1925, p. 405; The Work of Oliver Heaviside, El. J., **25,** 1928, p. 26. Other references will be given at points where they especially apply.

Now in Chapter III the steady-state response of a network was written in terms of the impedance function:

$$i_s(t) = \frac{1}{2j}\left(\frac{1}{Z(j\omega)}\,\epsilon^{j\omega t} - \frac{1}{Z(-j\omega)}\,\epsilon^{-j\omega t}\right). \tag{112}$$

Comparing the two expressions for $i_s(t)$, we can equate the coefficients of $\epsilon^{j\omega t}$ and obtain

$$\frac{1}{2jZ(j\omega)} = \frac{\omega}{2}\int_0^\infty \cos\omega\lambda A(\lambda)\,d\lambda + \frac{\omega}{2j}\int_0^\infty \sin\omega\lambda A(\lambda)\,d\lambda. \tag{113}$$

The two integrals may now be combined and converted to the exponential form, giving

$$\frac{1}{j\omega Z(j\omega)} = \int_0^\infty \epsilon^{-j\omega\lambda} A(\lambda)\,d\lambda. \tag{114}$$

This is a general relationship between Z and A, which theoretically at least, can always give A when Z is known. Equating coefficients of $\epsilon^{-j\omega\lambda}$,

$$\frac{1}{-j\omega Z(-j\omega)} = \int_0^\infty \epsilon^{j\omega\lambda} A(\lambda)\,d\lambda. \tag{114a}$$

We can replace $j\omega$ or $-j\omega$ in these expressions with any other variable, say a, provided only that the resulting infinite integral converges and defines a function. So

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\,d\lambda \tag{115}$$

which is the **infinite integral theorem.**

Our derivation here is limited by the condition governing (107), that the form of $A(t)$ be such that the integrals of that expression converge. Later (in Chapter X) we shall see that the infinite integral appears as a natural result of the Fourier integral formulation, and this limitation will there be modified.

Another condition on the form of $A(t)$ is evidently that it

be zero when $t < 0$, for the integral takes no account of any values of $A(t)$ for negative t. This is not serious, for any usual operator applied to the unit function yields a result which is also zero when $t < 0$. Later on operators will be met which are unusual in this respect, so it is well to remember the limitation.

3. **Illustrative Examples.** As a very simple example, consider the application of equation (114) to a circuit consisting of an inductance and resistance in series. The alternating-current impedance is

$$Z(j\omega) = R + jL\omega.$$

So

$$\frac{1}{j\omega(R + jL\omega)} = \int_0^\infty \epsilon^{-j\omega\lambda} A(\lambda)\, d\lambda. \tag{116}$$

Now this equation is satisfied by

$$A(\lambda) = \frac{1}{R}\left(1 - \epsilon^{-\frac{R\lambda}{L}}\right) \tag{117}$$

as can be proved by substituting in (116) and performing the integration. Therefore the indicial admittance of the network is

$$A(t) = \frac{1}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right). \tag{117a}$$

The equation thus furnishes a check upon the correctness of the known expression for A. Note, however, that it does not offer a ready means for finding A if it is unknown. To solve for A when Z is known, we are required to find an expression which will integrate into the known function. In any complicated case or even in simple cases, as a study of the above example will show, it is hardly possible to guess offhand at an expression that will do this; and, if guessing fails and we have no separate guide to what A may be like, there are no simple mathematical processes available by which it can be found.

Upon first realizing this situation, one is likely to jump to the conclusion that an equation which cannot be practically solved for an unknown is useless. On the contrary, it may be very valuable. In direct operational methods as practiced by Heaviside, the value of A was often readily found by processes that were not strictly rigorous, a very considerable doubt remaining as to whether the value found was correct or not. The integral theorem settles many such questions and therefore puts the operational scheme on a firmer basis.

4. **Geometric Interpretation.** It is usually best to use Carson's form[1] of the integral theorem:

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda. \tag{115}$$

This equation, like the superposition theorem from which it is derived, has a variable parameter under the sign. It is of interest to visualize the graphical meaning of the expression. The three-dimensional diagram of Fig. 20 has axes for λ, a, and y. In the y–λ plane is plotted

$$y = A(\lambda).$$

Through every point of this curve an exponential is drawn in a plane parallel to the y–a plane, the height of the exponential at any point being $A(\lambda_h)\epsilon^{-\lambda_h a}$ where λ_h is any definite value of λ. It will be noted that the decrement is proportional to the value of λ used; that is, the exponentials have a decay constant which increases the further they are spaced from the a–y plane.

These exponentials sweep out a surface. Several exponentials are drawn in the figure and indicate how this surface lies; but, if it is difficult to visualize it, this can be accomplished by actually constructing a three-dimension model of cardboard.

A section of the solid bounded by this surface and the a–λ

[1] Electric Circuit Theory and the Operational Calculus, p. 21.

plane is now taken by constructing a plane parallel to the λ–y plane. Such a section is shown. The area of this

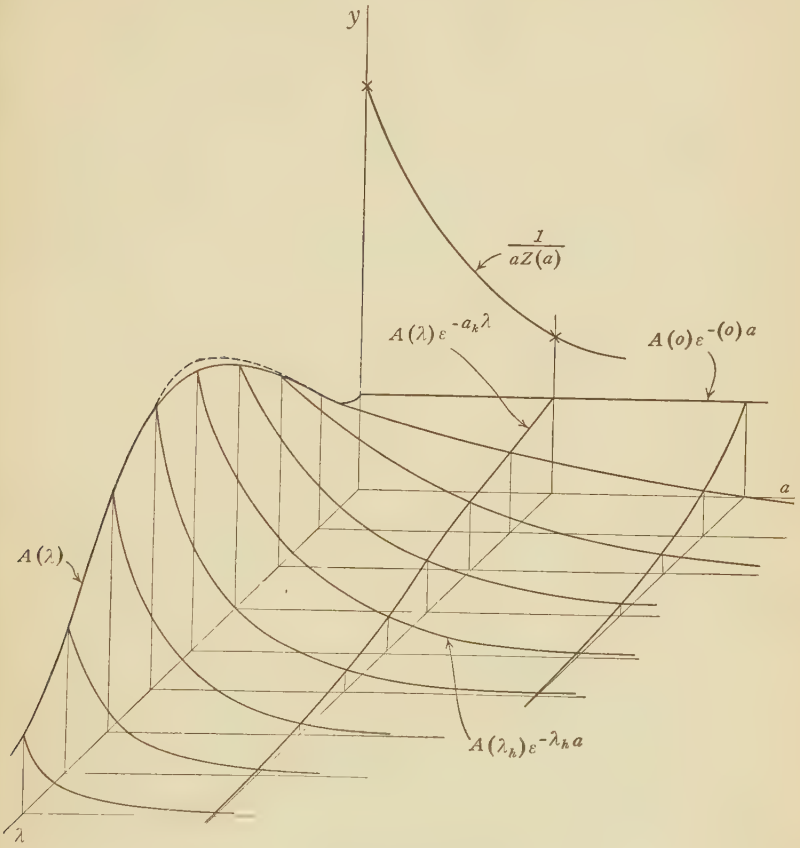

FIG. 20. Diagram of Equation (115).

section is the value of

$$\int_0^\infty \epsilon^{-a_k\lambda} A(\lambda)\, d\lambda$$

for a particular value of a. Plot the value of area against this value of a, as shown by the cross. A series of such con-

structions for various values of a will map out a curve:

$$y = \frac{1}{aZ(a)}$$

on the a–y plane.

This construction shows how, if A is known, the corresponding Z can be obtained graphically, this being inherently a three-dimensional process. The converse process is much harder. If Z is known and the curve in the y–a plane is plotted, then it is desired to so lay out a curve in the y–λ plane that, on carrying out this construction, we shall arrive at the curve we first plotted. Note that this cannot be done point by point. In the diagram above, if the $A(\lambda)$ curve be altered about one point, as shown by the dotted line, we shall change the entire $\frac{1}{aZ(a)}$ curve that results.

An excellent and not entirely profitless amusement will be found in devising mechanical, or hydraulic, or optical schemes for carrying out this graphical process automatically. Many such can be set up which will work beautifully — on paper.

5. **Form for Impulsive Network.** The integral theorem as written, since it was derived from the simple form of the superposition theorem, is similarly limited to cases in which $A(t)$ is not impulsive. For circuits of **impulsive response** a similarly modified integral theorem can be derived from the modified superposition theorem.

The superposition theorem for an impulsive network can be written:

$$i(t) = \frac{d}{dt}\int_0^t e(\lambda)\,\bar{A}(t-\lambda)\,d\lambda + \bar{C}e'(t) + \bar{C}e(0)p\,1. \quad (118)$$

For a sinusoidal impressed electromotive force,

$$e(t) = \sin \omega t.$$

The steady-state term can be derived by the same procedure as before, and becomes

$$i_s(t) = \omega \cos \omega t \int_0^\infty \cos \omega\lambda \bar{A}(\lambda)\, d\lambda + \omega \sin \omega t \int_0^\infty \sin \omega\lambda \bar{A}(\lambda)\, d\lambda + \bar{C}\omega \cos \omega t. \quad (119)$$

As before, this can be put in exponential form:

$$i_s(t) = \frac{\omega}{2}(\epsilon^{j\omega t} + \epsilon^{-j\omega t}) \int_0^\infty \cos \omega\lambda \bar{A}(\lambda)\, d\lambda + \frac{\omega}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) \int_0^\infty \sin \omega\lambda \bar{A}(\lambda)\, d\lambda + \frac{\bar{C}\omega}{2}(\epsilon^{j\omega t} + \epsilon^{-j\omega t}). \quad (120)$$

Now, upon comparing coefficients of $\epsilon^{j\omega t}$ with the expression for $i_s(t)$ in terms of Z, we have, on reducing,

$$\frac{1}{j\omega Z(j\omega)} = \int_0^\infty \epsilon^{-j\omega\lambda} \bar{A}(\lambda)\, d\lambda + \bar{C} \quad (121)$$

or, on making the change of variable,

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} \bar{A}(\lambda)\, d\lambda + \bar{C}. \quad (115a)$$

This may be regarded as a more general form, applicable to all linear networks whether impulsive or not. When the network is non-impulsive, $\bar{C} = 0$.

To illustrate, we shall apply this form to the simplest impulsive circuit — a condenser assumed to be without series resistance. The impedance is

$$Z(j\omega) = \frac{1}{j\bar{C}\omega}. \quad (122)$$

So

$$\frac{1}{aZ(a)} = C. \quad (123)$$

The equivalent capacitance of the condenser group is here simply $\bar{C} = C$. So the integral becomes

$$C = \int_0^\infty \epsilon^{-a\lambda} \bar{A}(\lambda)\, d\lambda + C \tag{124}$$

which is satisfied by $\bar{A}(\lambda) = 0$.

The response of the condenser to unit voltage is a current impulse corresponding to the capacitance C; that is, the instantaneous creation of a charge on the condenser plus zero current thereafter. Evidently the unmodified equation would have given an erroneous result if applied to such an impulsive network.

If the pure condenser is shunted by a leak of conductance G, the impedance is

$$Z(j\omega) = \frac{1}{jC\omega + G} \tag{125}$$

so that

$$\frac{1}{aZ(a)} = \frac{Ca + G}{a} = C + \frac{G}{a} \tag{126}$$

and the integral theorem gives

$$C + \frac{G}{a} = \int_0^\infty \epsilon^{-a\lambda} \bar{A}(\lambda)\, d\lambda + C. \tag{127}$$

This is satisfied by

$$\bar{A}(\lambda) = G. \tag{128}$$

The response now consists of the same impulse, followed by a steady current through the leak.

While in practical work impulsive circuits can usually be excluded from consideration, it is very convenient to have the modified form, so that in making operational transformations we do not need to make exceptions for each rule.

PROBLEMS ON CHAPTER VI

Prob. 1–6. The derivation of Sec. 2, Chapter VI can be made by putting all terms in sinusoidal form in place of exponentials. Show that this leads to the same result as obtained there.

Prob. 2–6. Find the indicial admittance of an R, L, and C series circuit and prove the result by means of the infinite integral theorem.

Prob. 3–6. Find the indicial admittance of the coupled circuit of Fig. 19, and prove the result by means of the infinite integral theorem.

Prob. 4–6. Draw the three-dimensional diagram like Fig. 20, for a simple series circuit having $R = 100$ ohms, and $L = 2\ h$. Check one or two points on the $\dfrac{1}{aZ(a)}$ curve by mechanical or graphical integration.

Prob. 5–6. Utilize the infinite integral theorem to prove the correctness of the expression for $A(t)$ obtained in Prob. 5–4.

CHAPTER VII

THE HEAVISIDE EXPANSION THEOREM

In his Electromagnetic Theory[1] Heaviside stated an important theorem by which the response of a linear network to a constant suddenly-applied electromotive force can be found in terms of the natural oscillations of the network. In our notation this so-called **expansion theorem** is

$$A(t) = \frac{1}{Z(0)} + \sum' \frac{\epsilon^{\lambda_k t}}{\lambda_k Z'(\lambda_k)} \tag{129}$$

in which $Z(\lambda)$ is the impedance function of the network and $Z'(\lambda)$ is its derivative with respect to λ. The summation indicates that we are to write as many terms as there are roots of the equation $Z(\lambda) = 0$, and into each insert the value of one of these roots.

The first term of the expression is the **steady-state** current, and the remainder is of exactly the form of the complementary function for the differential equations of the network, where the constants of integration are now replaced by expressions in terms of Z. Evidently it is possible to thus express the **amplitudes of oscillation** in terms of the impedance function of the network and the nature of the applied voltage, these amplitudes appearing in the form:

$$\frac{1}{\lambda_k Z'(\lambda_k)}.$$

A very simple example will show, to use a favorite expression of Heaviside, " the go of the matter." Consider a network consisting of series inductance and resistance only, to which a unit voltage is applied. Here

$$Z(\lambda) = R + L\lambda \tag{130}$$

[1] Vol. 2, p. 127.

and there is a single root of the determinantal equation:

$$\lambda_1 = -\frac{R}{L}. \tag{131}$$

It will be noted that

$$Z(0) = R$$

and that $$Z'(\lambda) = L$$

and $$\lambda Z'(\lambda) = L\lambda$$

so that $$\lambda_1 Z'(\lambda_1) = L\lambda_1 = -R.$$

The insertion of these in the expansion theorem gives

$$A(t) = \frac{1}{R} - \frac{1}{R}\epsilon^{-\frac{Rt}{L}}. \tag{132}$$

For lumped circuits where the number of roots involved is small, the solution is thus given with very little labor and with the use of extremely simple mathematics. When the number of roots is large, that is, when the network has several circuits, we run into trouble; not due to any fault of the theorem itself, but simply because of the difficulty of finding the roots.

The expansion theorem gives a **general relationship** between Z and A; but, unlike the integral theorem, it does so explicitly. As with the latter, we encounter practical difficulty when we use it to find an unknown A from a known Z; in this case because of the trouble involved in finding the roots of a higher-degree algebraic equation, especially when some or all of the roots are complex. The expansion theorem has proved serviceable in many practical problems.[1] It

[1] See, for example, W. H. Eccles, Application of Heaviside's Resistance Operators to the Air-core Transformer and Coupled Circuits in General, Proc. Phys. Soc. London, **24**, 1912, p. 260. Also V. Bush, Coupled Circuit by Method of Generalized Angular Velocities, Proc. I.R.E., 1917.

can even be applied in some cases to distributed circuits with an infinite series of roots.[1]

The form as given above applies when (1) the network is linear, (2) the resistances involved are positive, that is, there are no sources which might cause natural oscillations to build up in amplitude, (3) the roots of the determinantal equation are all distinct and different from zero,[2] (4) the response is not impulsive.

2. **Historical.** The expansion theorem has had a peculiar history. It was nearly forgotten and unused for many years, together with much of Heaviside's great work. On the other hand, it has occasionally been regarded as the subject of the whole operational calculus, whereas Heaviside himself correctly regarded it as simply one important theorem of that general subject.

It has been proved many times and in many ways. Heaviside demonstrated it twice, although this fact remained unrecognized by many, until Vallarta rearranged and presented the first proof and called attention to the second.[3] Bromwich was apparently one of the first to appreciate the value of the theorem, the proof of which he reviewed in a paper read in 1914,[4] but unfortunately not published until 1916. He treats the matter by contour integrals, and this same general method was independently and effectively used by Wagner.[5] We shall consider this method in Chapter

[1] For examples see E. Berg, Heaviside's Operators in Engineering and Physics, J. Franklin Inst., **198,** 1924, p. 647.

[2] A different form for the case of repeated root was developed by Wagner. See p. 91 of this text.

[3] M. S. Vallarta, Heaviside's Proof of his Expansion Theorem, Trans. A.I.E.E., **45,** 1926, p. 429. In the discussion of Vallarta's paper, J. J. Smith reviews the second proof, Trans. A.I.E.E., **45,** 1926, p. 433.

[4] T. J. I'A. Bromwich, Normal Coördinates in Dynamical Systems, Proc. London Math. Soc., **15,** 1916, pp. 401, 415–420.

[5] K. W. Wagner, Über eine Formel von Heaviside zur Berechnung von Einschaltvorgänge, Archiv f. Elek., **4,** 1916, p. 159. This proof is translated and appended to the I.R.E. paper referred to on p. 87.

IX. Carson gives two proofs, one of which is based on his infinite integral.[1] Several other investigators have recently discussed the subject.[2]

3. **Derivation by Expansion into Partial Fractions.** We shall give **Heaviside's second proof**[3] on account of its simplicity and directness, changing the form slightly to avoid the use of certain operational processes which were employed by Heaviside but which we have not yet substantiated.

The differential equation of the network is

$$i(t) = \frac{M(p)}{D(p)} e(t). \tag{133}$$

$M(p)$ may be written, if we please, $M_{hk}(p)$, signifying that the unit voltage is applied in the h circuit and the current evaluated in the k circuit, but we shall omit the subscripts and simply note that the expansion theorem may be used to evaluate either a driving point or a transfer indicial admittance.

We shall now use an **algebraic theorem**[4] which states that

$$\frac{f(x)}{F(x)} = \sum_{k=1}^{k=n} \frac{f(x_k)}{(x - x_k)F'(x_k)}. \tag{134}$$

$F(x)$ and $f(x)$ are polynomials in x, F being of higher degree than f; and $x_1, x_2, \ldots x_n$ are roots of $F(x)$ which are different from zero and no two of which are equal.

[1] J. R. Carson, A General Expansion Theorem, Phys. Rev., **10,** 1917, p. 217; Theory of the Transient Oscillations of Electrical Networks, Trans. A.I.E.E., *loc. cit.*; Electric Circuit Theory and the Operational Calculus; B.S.T.J., **4,** 1925, p. 685; B.S.T.J., **5,** 1926, pp. 50, 336; see also Carson's book, *loc. cit.*

[2] Mochinori Gotō, Extension of the Heaviside Expansion Theorem, Researches of the Electrotechnical Lab., Tokyo, Japan, No. 201.

[3] Electrical Papers, Vol. 2, p. 226. The footnote contains the proof and supersedes the text. L. Cohen gives a modification of Heaviside's second proof, which, however, also employs operational processes we have not yet substantiated. See L. Cohen, The Heaviside Expansion Theorem, J. Franklin Inst., **194,** 1922, p. 765.

[4] Consult, for example, B. Williamson, Integral Calculus. Longmans, 1896, p. 42.

If (134) be applied to (133),

$$\frac{M(p)}{D(p)} = \sum_{k=1}^{k=n} \frac{M(\lambda_k)}{(p - \lambda_k) D'(\lambda_k)} \tag{135}$$

and

$$i(t) = \sum \frac{M(\lambda_k)}{D'(\lambda_k)} \frac{1}{p - \lambda_k} e(t). \tag{136}$$

Here $\lambda_1, \lambda_2, \ldots \lambda_n$ are the roots of the determinantal equation, $D(\lambda) = 0$. It should be remembered that (136) is valid only when M and D are polynomials, when M is of lower degree than D, and when there are no repeated roots. Therefore the expression $M(p)/D(p)$ is to be reduced to a proper fraction before applying the algebraic theorem. No confusion will be caused by retaining the same symbols for the expression after this change is made.

Each term of (136) yields a linear differential equation, which could be written

$$\frac{d}{dt} i_k(t) - \lambda_k i_k(t) = \frac{M(\lambda_k)}{D'(\lambda_k)} e(t). \tag{137}$$

We shall now consider $e(t)$ to be the unit applied electromotive force. This equation is exactly the form of equation obtained when a unit e.m.f. is suddenly applied to an inductive circuit. When λ_k is a negative real, it is in fact exactly the same equation with the same boundary conditions. The well-known solution for such an equation, with these boundary conditions, is

$$i_k(t) = \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)} (\epsilon^{\lambda_k t} - 1). \tag{138}$$

When λ is complex but with a negative real portion, the physical analogy does not hold but the same mathematical steps of solving the equation give the same results. So the total current equation, for unit applied e.m.f. is

$$A(t) = \sum \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)} \epsilon^{\lambda_k t} - \sum \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)}. \tag{139}$$

Looking back at the expansion, we see that the last term of this equation is

$$\frac{M(0)}{D(0)}.$$

So

$$A(t) = \frac{M(0)}{D(0)} + \sum \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)} \epsilon^{\lambda_k t}. \tag{140}$$

This is one useful form as it stands. However,

$$Z(\lambda) = \frac{D(\lambda)}{M(\lambda)}. \tag{141}$$

So

$$Z'(\lambda) = \frac{M(\lambda)\, D'(\lambda) - D(\lambda)\, M'(\lambda)}{[M(\lambda)]^2} \tag{142}$$

and, noting that $D(\lambda_k) = 0$,

$$Z'(\lambda_k) = \frac{D'(\lambda_k)}{M(\lambda_k)}. \tag{143}$$

Inserting this in the expression above gives the usual form:

$$A(t) = \frac{1}{Z(0)} + \sum \frac{\epsilon^{\lambda_k t}}{\lambda_k Z'(\lambda_k)}. \tag{129}$$

This form of the expansion theorem applies to a linear dissipative network of lumped parameters when the response is not impulsive and when there are no coincident roots. The derivation as made here has also introduced the limitation that $M(\lambda)$ be of lower power than $D(\lambda)$. This limitation will be removed when the theorem is derived in Chapter IX by use of contour integrals. When $Z(\lambda)$ has multiple roots, Wagner[1] has derived an altered form which appears as follows: Suppose the roots are λ_1, λ_2, λ_3 and also λ_m which occurs m times. Then

$$A(t) = \frac{1}{Z(0)} + \sum \frac{\epsilon^{\lambda_k t}}{\lambda_k Z'(\lambda_k)} + \epsilon^{\lambda_m t}\left(B_1 + B_2 t + \cdots + B_m \frac{t^{m-1}}{(m-1)!}\right) \tag{144}$$

[1] *loc. cit.*

where the B's are constants, which may also be expressed in terms of Z, but which are usually more conveniently determined directly from the initial conditions.

Guillemin[1] has made a study of cases where the coefficients of some terms become zero. These cases are important in filters and corrective networks. This matter will be treated briefly in Sec. 9.

When the response is impulsive, we may split the network into parts and add a term in the manner discussed above in connection with the superposition theorem and the integral theorem.

4. **Another Form and Its Application to the Coupled Circuit.** It is sometimes convenient to have the equation in a **form not involving the derivative**, so that all computation involved in its use is algebraic. We may write

$$D(\lambda) = b(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_{2n}) \qquad (145)$$

where b is the coefficient of the highest power of λ in $D(\lambda)$. From this we can form the derivative:

$$D'(\lambda_k) = b(\lambda_k - \lambda_1)(\lambda_k - \lambda_2) \ldots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \ldots (\lambda_k - \lambda_{2n}). \qquad (146)$$

Insert this above and we have

$$A(t) = \frac{M(0)}{D(0)} + \sum \frac{M(\lambda_k)}{b\lambda_k(\lambda_k - \lambda_1)(\lambda_k - \lambda_2) \ldots} \epsilon^{\lambda_k t} \qquad (147)$$

where for each root there is a term in the summation which includes the product of all factors obtained by taking the difference between that root and each of the others.

Let us apply this form to the case of the **coupled circuit or air-core transformer** of Fig. 21. The differential equations are

$$\left.\begin{aligned} (R_1 + L_1 p)\, i_1 - L_{12} p i_2 &= E \\ -L_{21} p i_1 + (R_2 + L_2 p)\, i_2 &= 0. \end{aligned}\right\} \qquad (148)$$

[1] Making Normal Coördinates Coincide with the Meshes of an Electrical Network, Proc. I.R.E., **15**, 1927, p. 935.

So

$$D(\lambda) = \begin{vmatrix} (R_1 + L_1\lambda) & -L_{12}\lambda \\ -L_{21}\lambda & (R_2 + L_2\lambda) \end{vmatrix} \tag{149}$$

or

$$D(\lambda) = (R_1 + L_1\lambda)(R_2 + L_2\lambda) - L^2_{12}\lambda^2 \tag{150}$$

$$M_{11}(\lambda) = R_2 + L_2\lambda. \tag{151}$$

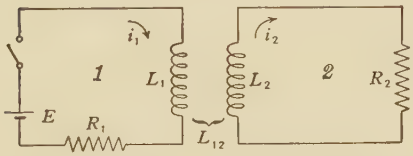

FIG. 21. Transformer Network.

The equation $D(\lambda) = 0$

is $$(L_1L_2 - L^2_{12})\lambda^2 + (R_1L_2 + R_2L_1)\lambda + R_1R_2 = 0 \tag{152}$$

which gives the two roots λ_1 and λ_2:

$$-\frac{R_1L_2 + R_2L_1}{2(L_1L_2 - L^2_{12})} \pm \sqrt{\left(\frac{R_1L_2 + R_2L_1}{2(L_1L_2 - L^2_{12})}\right)^2 - \frac{R_1R_2}{L_1L_2 - L^2_{12}}}. \tag{153}$$

Consider a simple numerical case:

$$R_1 = R_2 = 1 \text{ ohm},$$
$$L_1 = L_2 = 0.1 \text{ henry},$$
$$L_{12} = .05 \text{ henry}.$$

Then

$$\lambda_1 = -6.7 \qquad b = L_1L_2 - L^2_{12} = 0.0075$$
$$\lambda_2 = -20.$$

The difference of the roots is

$$\lambda_1 - \lambda_2 = 13.3.$$

Inserting these values,

$$A(t) = 1 + \frac{1 - 0.1 \times 6.7}{(-6.7 \times 13.3)\ .0075}\epsilon^{-6.7t} + \frac{1 - 0.1 \times 20}{(-20 \times -13.3)\ .0075}\epsilon^{-20t}$$
$$= 1 - 0.5\,\epsilon^{-6.7t} - 0.5\,\epsilon^{-20t}.$$

If we had used the usual form, we would have formed

$$Z(\lambda) = R_1 + L_1\lambda - \frac{L^2_{12}\lambda^2}{R_2 + L_2\lambda}$$

$$\frac{dZ}{d\lambda} = L_1 - \frac{2\,(R_2 + L_2\lambda)\,L^2_{12}\lambda - L^2_{12}L_2\lambda^2}{(R_2 + L_2\lambda)^2}$$

and inserting our roots, have obtained

$$\frac{1}{\lambda_1 Z'(\lambda_1)} = -\frac{1}{2} \qquad \frac{1}{\lambda_2 Z'(\lambda_2)} = -\frac{1}{2}$$

which, inserted, give the same expression as before.

5. **Approximate Method for Highly-oscillatory Network.** Great labor is involved in applying the expansion theorem to lumped networks of more than two or three degrees of freedom. The difficulty lies in the solution of the determinantal equation when it is above the third degree, especially if roots are complex. With distributed networks the solution is usually impossible. This situation is relieved somewhat by **special schemes** for obtaining the roots;[1] but even so, the use of the expansion theorem is rather limited.

When the network is **highly oscillatory**, that is, when the resistances play a small part, approximate relations may be very convenient. We shall follow Guillemin,[2] but with a somewhat different presentation.

[1] W. V. Lyon, Note on a Method of Evaluating the Complex Roots of a Quartic Equation, J. Math. and Phys., **3**, 1924.

L. F. Woodruff, Note on a Method of Evaluating the Complex Roots of Sixth and Higher Order Equations, J. Math. and Phys., **4**, 1925, p. 164.

Y. H. Ku, Note on a Method of Evaluating the Complex Roots of a Quartic Equation, J. Math. and Phys., **5**, 1925.

The above are also published as M.I.T., E.E. Research Bulletins, Nos. 42, 48, and 53. They are summarized in Woodruff, Electric Transmission, *loc. cit.*, p. 228.

[2] E. A. Guillemin, Approximate Solution for Electrical Networks, Trans. A.I.E.E., **47**, 1928, p. 361.

A method for the case of the coupled circuit only is given by V. Bush, Coupled Circuits by Method of Generalized Angular Velocities, *loc. cit.*

Write the determinant of the system in its polynomial form:

$$D(\lambda) = A_n\lambda^n + A_{n-1}\lambda^{n-1} + \cdots + A_0. \tag{154}$$

We shall split this into two parts for convenience and include all terms of even powers in one part, which we shall call $H(\lambda)$, and all terms of odd powers in the second, which we call $G(\lambda)$. If n is even,

$$\left.\begin{aligned} H(\lambda) &= A_n\lambda^n + A_{n-2}\lambda^{n-2} + \cdots + A_0 \\ G(\lambda) &= A_{n-1}\lambda^{n-1} + A_{n-3}\lambda^{n-3} + \cdots + A_1\lambda. \end{aligned}\right\} \tag{155}$$

The case where n is odd will be mentioned later.

Now suppose there is a root of D which is nearly a pure imaginary. Call this root

$$\lambda_k = \alpha_k + j\omega_k \tag{156}$$

where α_k will be a negative and $j\omega_k$ a pure imaginary and where we assume that α_k is small in absolute value compared with ω_k. If this be inserted, the terms of H will be nearly reals and those of G will be nearly pure imaginaries. The root must hence be such a value as will nearly make each of these independently equal to zero. Therefore if we solve

$$H(\lambda) = 0 \tag{157}$$

for the corresponding root, which we may call $j\bar{\omega}_k$, the value of $j\bar{\omega}_k$ will not be far different from that of $j\omega_k$. This is a great help, for $H(\lambda) = 0$ can be solved for λ^2, and it is then of only half the degree of $D(\lambda) = 0$. Thus the approximate angular velocities for a network involving a sixth-degree equation are obtained by solving a cubic. An approximate solution will then be

$$\bar{\lambda}_k = \bar{\alpha}_k + j\bar{\omega}_k \tag{158}$$

and we shall proceed to find $\bar{\alpha}_k$. Substitute $\bar{\lambda}_k$ into $D(\lambda)$ and in expanding each term save only those terms which do not contain $\bar{\alpha}_k$, or which are multiplied by $\bar{\alpha}_k$ to the first

power, on the assumption that terms containing higher powers of $\bar{\alpha}_k$ are negligible in comparison.

$$A_n[(j\bar{\omega}_k)^n + n(j\bar{\omega}_k)^{n-1}\bar{\alpha}_k] + A_{n-1}[(j\bar{\omega}_k)^{n-1} + (n-1)(j\bar{\omega}_k)^{n-2}\bar{\alpha}_k] + \cdots + A_0 = 0. \tag{159}$$

This may be written

$$H(j\bar{\omega}_k) + \bar{\alpha}_k\left(\frac{d}{d\lambda}H(\lambda)\right)_{\lambda=j\bar{\omega}_k} + \bar{\alpha}_k\left(\frac{d}{d\lambda}G(\lambda)\right)_{\lambda=j\bar{\omega}_k} + G(j\bar{\omega}_k) = 0. \tag{160}$$

The first term is zero, the third is real, and the other two are pure imaginary; so the third term is zero and

$$\bar{\alpha}_k = -\frac{G(j\bar{\omega}_k)}{H'(j\bar{\omega}_k)}. \tag{161}$$

We can thus rapidly arrive at values for the natural frequencies and decrements. Furthermore, in forming $H(\lambda)$ the coefficients will be only slightly altered if the resistances are set equal to zero, while with similar treatment the coefficients of $G(\lambda)$ will vanish. In order to obtain the approximate angular velocities, we may solve

$$\overline{D}(\lambda) = 0$$

where $\overline{D}$ is formed by placing all resistances equal to zero in the network. This last step only slightly abbreviates the work.

We can write a form of the expansion theorem which is useful in highly-oscillatory networks of this sort. In the form:

$$A(t) = \frac{M(0)}{D(0)} + \sum \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)} \epsilon^{\lambda_k t} \tag{162}$$

we may replace λ_k approximately by $j\bar{\omega}_k$ in the coefficient of the second term. Also it is known that $G'(j\bar{\omega}_k) = 0$. So approximately,

$$A(t) = \frac{M(0)}{D(0)} + \sum \frac{\overline{M}(j\bar{\omega}_k)}{j\bar{\omega}_k H'(j\bar{\omega}_k)} \epsilon^{\bar{\lambda}_k t}. \tag{163}$$

This form is convenient, for $H'(j\bar{\omega}_k)$ has already been used in finding $\bar{\alpha}_k$.

It was assumed that n was even. If n is odd, there will be one real root; and if the other roots correspond to a highly-oscillatory condition, we can divide out of $D(\lambda)$ the factor for the real root and then proceed as before. An approximate value for the real root is $-A_0/A_1$. When all other roots are nearly pure imaginaries this is usually a very good approximation, for A_2 will be small.

In brief, this method consists of finding the natural angular velocities of oscillation by assuming all resistances zero. The decrements are then found by an approximate formula in terms of these angular velocities. When the resistances are relatively small the results so obtained will very closely approximate the true values.

6. **Relation to Infinite Integral Theorem.** Since both the Heaviside expansion theorem and the infinite integral theorem express a relation between the functions Z and A, it should be possible to convert one into the other. Carson in fact derives the expansion theorem from the integral theorem in this manner. It will be of interest to proceed in the other direction to show the relationship; noting, however, that the expansion theorem requires that there be no coincident roots while the integral theorem as we obtained it above is not so limited. The following is not valuable as a derivation, therefore, but is presented rather to show the interrelationship.

Into the integral:

$$\int_0^\infty \epsilon^{-\lambda t} A(t)\, dt \tag{164}$$

insert the expression for $A(t)$ from the first form of the expansion theorem, and there results

$$\frac{M(0)}{D(0)} \int_0^\infty \epsilon^{-\lambda t}\, dt + \sum \frac{M(\lambda_k)}{\lambda_k D'(\lambda_k)} \int_0^\infty \epsilon^{(\lambda_k - \lambda)t}\, dt \tag{165}$$

which becomes, if λ has a positive real portion,

$$\frac{M(0)}{\lambda D(0)} + \sum \frac{M(\lambda_k)}{\lambda_k(\lambda - \lambda_k) D'(\lambda_k)}. \tag{166}$$

Referring back to the expansion (134), this is seen to be

$$\frac{M(\lambda)}{\lambda D(\lambda)}$$

for, on expanding this expression, one of the roots of the denominator, $\lambda D(\lambda)$, is $\lambda = 0$. We have thus shown, under the conditions stated, that

$$\frac{1}{\lambda Z(\lambda)} = \int_0^\infty \epsilon^{-\lambda t} A(t)\, dt. \tag{167}$$

λ in this expression is simply a positive real quantity, or a complex with positive real portion great enough so that the integrated expression vanishes at the upper limit. This, of course, expresses exactly the same relationship as

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda. \tag{115}$$

7. **Form for Sinusoidal Electromotive Force.** The expansion theorem is written for a unit applied e.m.f. It may also be written for other forms of applied e.m.f. by inserting the usual expression into the superposition theorem. Carson[1] does this for a sinusoidal applied voltage, one term of which is $E\epsilon^{j\omega t}$. The derivation is straightforward and need not be reviewed here, as this form will be again encountered later on. This gives an expression:

$$i(t) = \frac{E\epsilon^{j\omega t}}{Z(j\omega)} + \sum \frac{E\epsilon^{\lambda_k t}}{(j\omega - \lambda_k) Z'(\lambda_k)}. \tag{168}$$

This form is interesting especially in connection with cases of the application of sinusoidal voltage to a network where

[1] Phys. Rev., *loc. cit.*, p. 224. Cohen, *loc. cit.* also derives such an expression operationally using, however, operational expressions not yet established.

the frequency of the applied voltage is close to one of the natural frequencies of the network. In that case not only is the steady-state amplitude large, for $Z(j\omega)$ is small, but the particular transient term corresponding to this frequency is also large.

Usually, however, in studying networks under various forms of applied voltage, it is best to proceed in two steps — evaluate $A(t)$ and then use the superposition theorem.

8. **Use on a Distributed Network.** We shall illustrate the use of the expansion theorem by the problem of a cable of resistance r and capacitance c per unit length, the other parameters being negligible. Our proof of the theorem is limited to the lumped network, and we must regard its use here on a distributed network as justified only by the fact that the result we shall obtain can be checked by other operational processes which are not so limited. Consider the cable to be of total length D, grounded at the distant end and with an electromotive force E_0 suddenly applied at the home end. We shall solve for the voltage e at a point x units from the home end.

In the alternating-current steady state, the ratio of these voltages is given by a ratio of hyperbolic sines, thus

$$\frac{E_0}{E} = \frac{\sinh \beta D}{\sinh \beta (D - x)} \tag{169}$$

where

$\beta = \sqrt{jrc\omega}$ and E is the steady-state voltage at any point x. (170)

Hence the operational expression is

$$e = E_0 \frac{\sinh \beta (D - x)}{\sinh \beta D} 1 \tag{171}$$

where $$\beta = \sqrt{rcp}$$

and $$Z(p) = \frac{\sinh \beta D}{\sinh \beta (D - x)}. \tag{172}$$

The roots may be found by equating $Z(\lambda)$ to zero, which gives

$$\sinh D\sqrt{rc\lambda} = 0. \tag{173}$$

This is satisfied by an infinite series of values obtained from

$$D\sqrt{rc\lambda} = jk\pi \tag{174}$$

for $\sinh jk\pi$ is zero when k is any integer. So our series of roots is

$$\lambda_k = -\frac{k^2\pi^2}{rcD^2} \tag{175}$$

$$k = 1, 2, \ldots .$$

We need $\frac{1}{Z(0)}$, and since both numerator and denominator of $Z(\lambda)$ are zero for $\lambda = 0$ we obtain the result by the usual process of evaluating indeterminate forms:

$$\frac{1}{Z(0)} = \lim_{\beta \to 0} \frac{\sinh \beta(D - x)}{\sinh \beta D}$$

$$= \lim_{\beta \to 0} \frac{(D - x)\cosh \beta(D - x)}{D \cosh \beta D} = 1 - \frac{x}{D}. \tag{176}$$

This could have been predicted by evaluating the steady-state term of our result by inspection. Now $\lambda_k Z'(\lambda_k)$ is needed. To obtain this we shall take the derivative of $Z(\lambda)$ above, substitute $\lambda = \lambda_k$, and in the process drop out the term $\sinh D\sqrt{rc\lambda_k}$ which is known to be zero.

$$\lambda_k Z'(\lambda_k) = \frac{D\beta_k \cosh \beta_k D}{2 \sinh \beta_k(D - x)} \tag{177}$$

and since $\cosh jk\pi = \cos k\pi = \mp 1$,

$$\sinh\left(jk\pi - \frac{jk\pi x}{D}\right) = j \sin k\pi\left(1 - \frac{x}{D}\right) = \pm j \sin \frac{k\pi x}{D}$$

$$\lambda_k Z'(\lambda_k) = -\frac{k\pi}{2 \sin \dfrac{k\pi x}{D}}. \tag{178}$$

Inserting these various values in the expansion theorem:

$$e = E_0\left(1 - \frac{x}{D}\right) - \frac{2E_0}{\pi}\sum\frac{1}{k}\sin\frac{k\pi x}{D}\,\epsilon^{-\frac{k^2\pi^2 t}{rcD^2}}. \qquad (179)$$

The current may now be obtained from the relation:

$$i = -\frac{1}{r}\frac{\partial e}{\partial x}$$

$$i = \frac{E_0}{rD} + \frac{2E_0}{rD}\sum\cos\frac{k\pi x}{D}\,\epsilon^{-\frac{k^2\pi^2 t}{rcD^2}}. \qquad (180)$$

The arrival current is obtained by putting $x = D$ into this expression.

$$i_D = \frac{E_0}{rD} - \frac{2E_0}{rD}\left(\epsilon^{-\frac{\pi^2 t}{rcD^2}} - \epsilon^{-\frac{4\pi^2 t}{rcD^2}} + \cdots\right). \qquad (181)$$

A great many results of this general sort have been evaluated.[1] They are not as valuable for cable work as they once were; for the neglection of inductance is serious when high speed is considered, and in modern loaded cables inductance is all-important.[2] However, the same formulas are as important as ever in connection with the flow of heat.

By combining results such as the above, the form of arrival current may be obtained when the sending voltage is made up of a series of telegraph impulses. Figure 22 shows a case of this sort for illustration.

The expansion theorem gives its results in the form of a series of exponentials. For a lumped network where these represent the natural modes of oscillation this is excellent. For a cable, and still more for a transmission line, this form of expression, even when it can be obtained, is not nearly so convenient as other forms. In the case of the transmission line the natural mode of subsidence is by traveling waves,

[1] See H. W. Malcolm, Theory of the Submarine Telegraph and Telephone Cable, London Electrician, Publishers, 1917.

[2] See, for instance, O. E. Buckley, The Loaded Submarine Telegraph Cable, B.S.T.J., **4**, 1925, p. 355; High-Speed Ocean Cable Telegraphy, B.S.T.J., **7**, 1928, p. 225.

and an attempt to analyze in other terms is bound to be difficult. The expansion theorem is not a natural tool for use on wave problems.

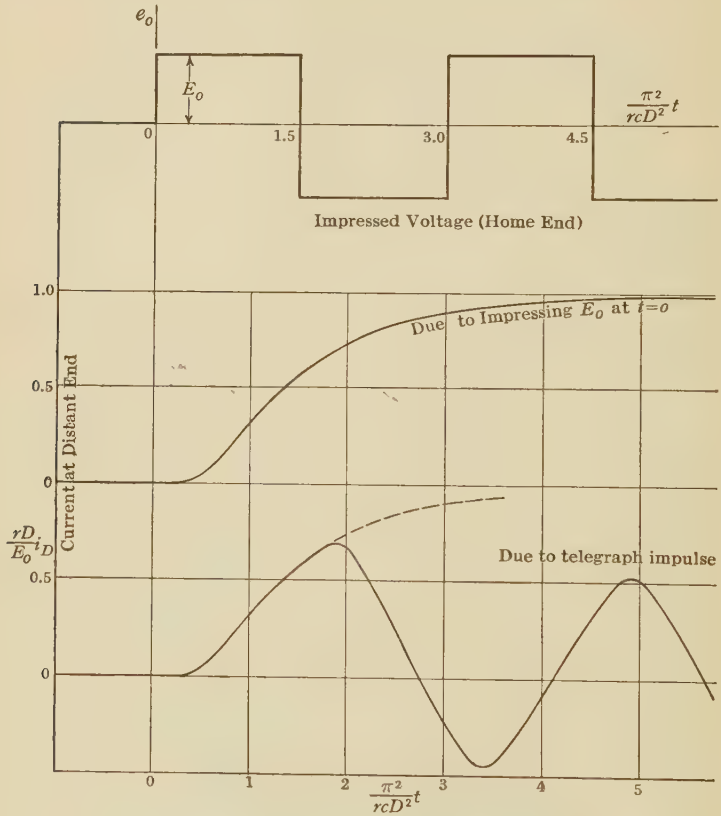

FIG. 22. Arrival Current of a Telegraph Cable.

9. **Isolated Modes.** It has been stated above that in general the same natural frequencies and decrements appear in the transient of a network, no matter what may be the nature of the applied force and no matter to what part of the network it may be applied, and that in general the same exponential terms appear in all the solutions for currents or

voltages in all parts of the network. This is generally true, for if there is a term of one generalized angular velocity at some point in the network it will induce similar terms throughout. Yet there are exceptions to this general rule, and **it is possible to have networks in which one normal mode of oscillation or subsidence is confined to a particular circuit.** This can occur when the total coupling between this circuit and the others goes to zero for the generalized angular velocity belonging to that circuit. It is then evident that this mode of oscillation can be aroused only when the disturbing force is applied to that particular circuit, and that it will appear in that circuit alone. It will be entirely absent from the transient when the disturbing force is applied elsewhere. Guillemin[1] has treated this matter and brings out clearly the conditions necessary for thus localizing a term. If a certain exponential term does not appear in the indicial admittance of the network when the unit voltage is applied at a certain point, it is evident from the superposition theorem that the term will also be absent when any other form of voltage is applied at the same point. We are thus led to examine

$$A(t) = \frac{M(0)}{D(0)} + \sum \frac{M_{gh}(\lambda_k)}{\lambda_k D'(\lambda_k)} \epsilon^{\lambda_k t} \tag{140}$$

where the subscripts in M_{gh} indicate that unit voltage is applied in Circuit g and the current evaluated in Circuit h. Evidently the conditions for the absence of the term for a particular angular velocity λ_r in the result is

$$M_{gh}(\lambda_r) = 0.$$

Now D is the determinant:

$$\begin{vmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{vmatrix} \tag{182}$$

[1] Proc. I.R.E., 1927, *loc. cit.*

and M_{gh} is the minor obtained by striking out the row and column containing z_{gh}:

$$\begin{vmatrix} z_{11} & z_{12} \cdots z_{1,h-1} & z_{1,h+1} \cdots z_{1n} \\ \cdots & \cdots & \cdots \\ z_{g-1,1} & z_{g-1,2} \cdots z_{g-1,h-1} & z_{g-1,h+1} \cdots z_{g-1,n} \\ z_{g+1,1} & z_{g+1,2} \cdots z_{g+1,h-1} & z_{g+1,h+1} \cdots z_{g+1,n} \\ \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} \cdots & z_{nn} \end{vmatrix} \tag{183}$$

Suppose that in the determinant of the system one root, say λ_r, is such that an entire column such as the h column vanishes; that is, that $z_{1h}, z_{2h} \ldots z_{nh}$ are all zero for this value λ_r. Since $z_{1h} = z_{h1}$, $z_{2h} = z_{h2}$, etc. it then follows that the h row will also vanish for this value λ_r. All the minors of the determinant will thus be zero for this particular root λ_r with the single exception of M_{hh}.

Looking back at our expansion theorem it thus appears that when the above condition holds, the indicial admittance will contain the term $\epsilon^{\lambda_r t}$ only when we examine a driving point indicial admittance examined for Circuit h. In other words the transient of the network will contain this particular natural frequency with its accompanying decrement only when excited by applying the voltage in the circuit to which that angular velocity belongs and not otherwise. When an oscillation of this frequency is thus set up, it is confined to that particular circuit and in no way can it be made to appear as a transient term elsewhere in the network. Other natural frequencies, unless the condition holds for some other root also, are transformed from circuit to circuit and appear throughout the network, but this particular one is confined. It is natural to expect that the value of the confined natural frequency depends only on the parameters of the mesh in which it is localized, and this we shall see is correct.

Expand the determinant D in terms of its minors of the h column:

$$D(\lambda) = z_{1h}M_{1h} + z_{2h}M_{2h} + \cdots + z_{nh}M_{nh}. \quad (184)$$

Now suppose each of these values of z can be obtained from z_{hh} by multiplying by a constant, so that $z_{1h} = b_1 z_{hh}$, $z_{2h} = b_2 z_{hh}$, etc. The determinant can then be written

$$D(\lambda) = z_{hh}(b_1 M_{1h} + b_2 M_{2h} + \cdots + b_n M_{nh}). \quad (185)$$

and the solution of $D(\lambda) = 0$ for the natural angular velocities gives one root from $z_{hh} = 0$. This is the root λ_r, for it makes each term in the h column of the determinant vanish. It is the natural frequency of the h Circuit alone, considered entirely apart from the rest of the network.

A network can be set up readily in which all the natural frequencies but one are confined to their respective circuits, and this one appears throughout the network. It should be emphasized, however, that the analysis of this problem of confined frequencies which has been made so far in terms of the expansion theorem is limited to the condition that no coincident roots appear from the solution of the determinantal equation.

This analysis is evidently important in connection with all sorts of selective networks, such as the tuned circuits and filters of communication networks. Such networks are often computed only for their steady-state response, whereas their behavior in the transient state is equally important. It is often desirable to isolate or confine undesired frequencies. As an example consider the mechanical networks of acoustic apparatus such as the phonograph. It has been possible[1] by steady-state analysis of these networks, using the analogies we have illustrated, to make considerable advance in phonograph design. Yet the more difficult transient analysis may be similarly productive, and the isolation of undesired frequencies is one of the real problems.

[1] Maxfield and Harrison, *loc. cit.*

We may well examine more in detail the meaning of our criterion for the isolation of a natural frequency. Consider a network of two circuits only, and let these be coupled in all three ways (Fig. 23). The impedance z_{11} is

$$z_{11}(\lambda) = R_{11} + L_{11}\lambda + \frac{1}{C_{11}\lambda}. \tag{186}$$

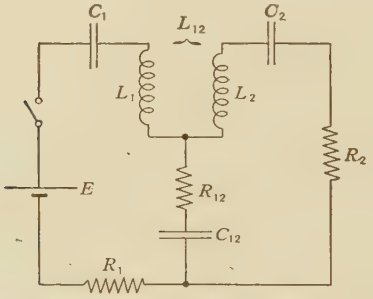

FIG. 23. Network with Three Types of Coupling.

If there is to be an isolation of frequencies,

$$z_{12}(\lambda) = Kz_{11}(\lambda) \tag{187}$$

where K is some constant. That is,

$$z_{12}(\lambda) = KR_{11} + KL_{11}\lambda + \frac{K}{C_{11}\lambda} \tag{188}$$

which simply means that the coupling factor K between the two circuits must have the same value for each type of coupling. Evidently this reasoning can readily be extended to any number of circuits.

When the network is highly oscillatory it is easy to see also by means of vector diagrams how this process of isolating a particular frequency is brought about due to a proper choice of coupling coefficients. The treatment above shows how this isolation can be accomplished in the general case by employing all three coupling coefficients, whatever the form of the natural modes of the network. When the transient term to be isolated is a simple exponential, two coupling

coefficients suffice; but when the term results from the combination of two exponentials with real or complex exponents, the employment of all three coupling coefficients becomes necessary. This will be brought out if we examine a network of two circuits with no condenser present in Circuit *1* (Fig. 24). Then

$$z_{11}(\lambda) = R_{11} + L_{11}\lambda \tag{189}$$

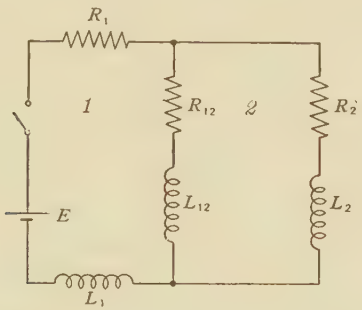

FIG. 24. Network with Isolated Mode.

and for frequency isolation,

$$z_{12}(\lambda) = K_1R_{11} + K_1L_{11}\lambda. \tag{190}$$

The secondary may contain a condenser and may be linked with other circuits. We shall consider for simplicity, however,

$$z_{22}(\lambda) = R_{22} + L_{22}\lambda \tag{191}$$

and shall not confine the second generalized frequency. Notice that the coupling may be as close or as loose as we please, provided K is the same for all types involved.

Let us form the determinant for this network:

$$D(\lambda) = \begin{vmatrix} R_{11} + L_{11}\lambda & -K_1R_{11} - K_1L_{11}\lambda \\ -K_1R_{11} - K_1L_{11}\lambda & R_{22} + L_{22}\lambda \end{vmatrix} \tag{192}$$

Placing this equal to zero, it may be written

$$(R_{11} + L_{11}\lambda)\,(R_{22} + L_{22}\lambda - K_1^2R_{11} - K_1^2L_{11}\lambda) = 0 \tag{193}$$

so that one root is

$$\lambda_1 = -\frac{R_{11}}{L_{11}} \tag{194}$$

and the other $$\lambda_2 = -\frac{R_{22} - K_1^2 R_{11}}{L_{22} - K_1^2 L_{11}}. \tag{195}$$

The minor for Circuit *1* is

$$M_{11}(\lambda) = R_{22} + L_{22}\lambda \tag{196}$$

and $$M_{11}(\lambda_1) = R_{22} - \frac{R_{11}L_{22}}{L_{11}}. \tag{197}$$

Similarly, $$\lambda_1 D'(\lambda_1) = -R_{11}\left(R_{22} - R_{11}\frac{L_{22}}{L_{11}}\right). \tag{198}$$

Thus for the current in Circuit *1*,

$$A_{11}(t) = \frac{1}{\overline{R}} - \frac{1}{R_{11}}\epsilon^{-\frac{R_{11}t}{L_{11}}} + \frac{M_{11}(\lambda_2)}{\lambda_2 D'(\lambda_2)}\epsilon^{\lambda_2 t} \tag{199}$$

where $\overline{R}$ is the equivalent resistance:

$$\overline{R} = R_1 + \frac{R_{12}R_2}{R_{12} + R_2}$$

and where we have evaluated only the term in which we are principally interested. Notice that, as far as this term is concerned, the transient proceeds exactly as though the secondary were not present. Also we have

$$M_{12}(\lambda_1) = 0$$

since a term of the form $\epsilon^{-\frac{R_{11}t}{L_{11}}}$ does not appear in the secondary at all. The reason for this is evident if induced voltages are considered. When a current of this particular exponential form flows in the primary, the voltage induced in the secondary due to the conductive coupling is $\frac{R_{12}}{R_{11}}\epsilon^{-\frac{R_{11}t}{L_{11}}}$, and due to the inductive coupling, $-\frac{L_{12}}{L_{11}}\epsilon^{-\frac{R_{11}t}{L_{11}}}$. But since $R_{12} = K_1 R_{11}$ and $L_{12} = K_1 L_{11}$, these voltages are equal and

opposite for this particular exponential. For a network of n circuits, $(n-1)$ natural frequencies can be confined to their respective circuits. If we try to confine all n natural frequencies we encounter necessarily the case of coincident roots, and special treatment is required. Thus, in the example above, if

$$\left.\begin{aligned} R_{12} &= K_2 R_{22} \\ L_{12} &= K_2 L_{22} \end{aligned}\right\} \tag{200}$$

Wagner's form of the expansion theorem gives

$$A_{11}(t) = \frac{1}{R} - \frac{1}{R_{11}} \epsilon^{-\frac{R_{11}t}{L_{11}}} + \frac{K_1 K_2}{L_{11}(1 - K_1 K_2)} t \epsilon^{-\frac{R_{11}t}{L_{11}}} \tag{201}$$

and the simple argument in terms of exponentials is no longer good.

If we take a network of three circuits in which *2* is coupled to *1* and *3*, we can evidently confine frequencies in each of these latter; but if *1* and *3* are also coupled, we immediately encounter the case of coincident roots.

10. **Summary.** We have now studied the superposition theorem, the integral theorem, and the expansion theorem. These are the principal working tools by which we can develop the type of operational processes with which we are here concerned. They are closely interrelated. Each expresses important general relationships, but each is at a practical disability in many engineering problems. They serve, however, as a foundation for the direct operational method to which we shall now turn.

PROBLEMS ON CHAPTER VII

Prob. 1–7. Write the terms of the expansion theorem as applied to each of the networks of Figs. 1, 3, 4, 19.

Prob. 2–7. Use the expansion theorem to obtain a solution for Prob. 4–4.

Prob. 3–7. Use the approximate treatment for a highly-oscillatory network to obtain a solution for the network of Fig. 6. Consider the voltage $e(t)$ to be constant at 100 volts. The parameters are: $C_1 = 10$ mfd., $C_2 = 20$ mfd., $C_{12} = 1$ mfd., $R_1 = 100$ ohm, $R_2 = 200$ ohms, $L_1 = 10$ henries, $L_2 = 20$ henries.

Prob. 4–7. Solve the problem treated in Sec. 8, considering the cable to be open at the distant end.

Prob. 5–7. Draw a network of three circuits in which *1* and *2* are coupled, and also *2* and *3*. Choose constants so that Circuit *1* shall have a natural frequency of 1000 cycles per second, and so that this mode shall be isolated. Also fix the natural frequency of Circuit *3* to be 2000 cycles per second, and isolate this also.

Prob. 6–7. A current of the form $\epsilon^{-500,000\,t} \sin 3 \times 10^6\, t$ flows in an antenna circuit which is coupled to a second circuit conductively by having a one-ohm resistance in common. Specify a mutual inductance and mutual capacitance which might be added to prevent any current of this same form being induced in the second circuit. Draw the network. What voltage will now be induced in the second circuit if a sustained current $0.1 \sin 3 \times 10^6\, t$ is maintained in the first circuit?

Prob. 7–7. Write the Heaviside expansion theorem. Consider that this is to be applied to the problem of the sudden introduction of a current I into a network of known characteristics. Give the meaning, under these conditions, of each expression involved in the theorem.

CHAPTER VIII

THE DIRECT OPERATIONAL METHOD

The **Heaviside Operational Calculus** consists of the direct application of operators of the sort previously discussed to certain discontinuous operands.

The differential equation of a linear network to which an electromotive force $e(t)$ is suddenly applied may be written

$$i(t) = \frac{1}{Z(p)} e(t). \tag{202}$$

In the classic scheme of solution of such problems the general solution of this equation is first found for the part of the range over which $e(t)$ may be regarded as a continuous function with all necessary derivatives, and then the fact that $e(t)$ is actually discontinuous is introduced later by determining the constants of integration in accordance with the initial conditions. The solution thus proceeds in two steps, in the first one of which an operator $\frac{1}{Z(p)}$ is applied to a continuous operand $e(t)$, and in the second one of which the solution is retouched to account for the discontinuity at $t = 0$. In this process we may use operational processes, treat p algebraically, and so on; but in applying the operator to the operand, we involve only derivatives and integrals of a continuous function, and hence employ only well-recognized processes. The operators here are strictly derivative operators.

Heaviside combined these two steps into one. He applied his operator directly to the operand, which he recognized as discontinuous from the outset. Naturally this soon involved him in processes which were new, for derivative operators become meaningless when applied to functions at points where these latter become discontinuous. It was

therefore necessary to develop in one way or another the rules for application of the generating function, so that results would be yielded in accordance with solutions obtained from classic methods of solution. Heaviside derived his results formula by formula, and with only occasional reference, as in his expansion theorem, to any general scheme of defining his operations.

He concentrated his efforts on one type of applied e.m.f. — the unit e.m.f.; and this procedure is excellent, for the effect of other forms of applied e.m.f. is readily evaluated in terms of this primary result through the superposition theorem.

Many of his results he obtained heuristically, and some undoubtedly through a shrewd guess guided by a cultivated intuition and a physical grasp of the problem in hand. He seldom went wrong; which, in view of his astounding courage, is certainly remarkable.

Heaviside was perhaps the first great mathematical engineer. He was a mathematician, self taught but with an excellent knowledge of mathematical processes. In his writings, however, his attitude was usually that of the engineer and not of the mathematician or even of the mathematical physicist of his day. He had certain engineering problems before him and he proposed to get at the best possible solution in view of the mathematical processes available. The most powerful tool he found in circuit analysis was keen but largely unpolished. He nevertheless used it, knowing that he was exposed to errors, and guarding against them in any way that occurred to him.

We are in a much better situation to-day, and many steps he took without mathematical justification can now be modified, if necessary, and substantiated. Some are still at the stage where justification and rigor are possible to the mathematician but hardly to the engineer.

It must not be thought that there is an inherent fixed scheme of operations of the sort which we consider, and that its development mathematically is a process of discovery.

Rather it is nearer to invention. The mathematician does not ordinarily discover new mathematical processes in the sense that a chemist isolates a new chemical compound. Rather he is to be compared to the sculptor who carves out of a block of marble a figure which existed there in the block before he took his tools in hand, but which existed more vitally in the brain of the artist.

When a new concept arrives in mathematics it may be given various forms, and this is especially true of operational ideas. It must of course be so formed that it will yield results that are not trivial, and in our operational development this means that it must be able to solve our problems correctly. A sculptored figure must be able to stand up and bear its own weight. There is still left, in forming a new mathematical tool however, much liberty of action in regard to its exact form and the definitions on which it is to be based. Heaviside hardly attempted a general formulation at all. Later workers have approached from different angles, and should not expect their schemes to be identical except in the one point that they should all yield correct results. That scheme is usually best which gives rules of procedure most nearly like those in the existing body of mathematics. We shall ordinarily base our treatment on the definition of operators which is obtained by comparison with the Fourier integral scheme of classic treatment of circuits. We shall also however consider schemes that are based on integrals in the complex plane. For the generating functions usually met, these two schemes of definition yield the same rules of procedure and the same operational formulas. They will not do so, however, for any $f(p)$ put together at random without any necessary relation to a physically possible network.

A review of the history of the place of complex quantities in mathematics is enlightening in this connection. When such quantities were first encountered there was no inherent reason why they should have been expected to follow one set

of rules of combination rather than another. They were assigned rules which would be as near like those applying to real quantities as possible, and which would not lead to inconsistencies with the existing body of mathematics. It was then found that they were exceedingly useful. For one thing, their laws of combination were found to be the same as for sinusoids; and this led to many things that the electrical engineer for one would not like to do without.

We shall proceed to the discussion of direct operational methods, justifying our processes as completely as possible.

2. **Various Ways of Evaluating Operational Expressions.** It is necessary to interpret

$$\frac{1}{Z(p)}e(t)$$

where we are primarily interested in functions of t which are zero when $t < 0$, and especially in the operational expression:

$$\frac{1}{Z(p)}1.$$

Interpretation means a transformation of this expression to yield a function of t no longer involving p. Any such expression need be evaluated only once, provided, of course, that it is single-valued or that a complete evaluation is obtained. Thereafter, wherever that particular combination of operator and operand is met, it may be interpreted at once by referring to the previous result. We shall, therefore, gather together a **table of operational formulas**[1]

[1] A short table will be found in Appendix C. A much more complete table is given in "The Practical Application of the Fourier Integral," by G. A. Campbell, Int. Congress of Telephony and Telegraphy in Commemoration of Volta, Lake Como, Italy, September, 1927. See B.S.T.J., **7**, 1928, p. 639. The formulas are there given in the form of Fourier transforms. Most of them become immediately available as operational formulas by noting that the operand is $p1$ in our notation. Excellent practice may be obtained by substantiating some of these expressions by operational methods.

which will serve for the operational calculus in much the same way that a table of integrals serves the integral calculus. We shall also study transformation schemes by which operational formulas may be modified, split up, or combined. These occupy the same position that schemes such as integration by parts or change of variable occupy in the evaluation of integrals in integral calculus.

The evaluation of operational expressions of the type considered can be carried out by:

1. Comparison with classic solutions. This is always what is attempted in evaluating an operational expression which has been set up to cover a particular physical problem. Several ways of making this comparison may be noted. One is the direct solution of the differential equation and the insertion of initial and boundary conditions. This method of interpreting an operational expression has been used in evaluating $\frac{1}{p+\lambda}1$ when deriving the expansion theorem.

2. Processes involving the Fourier integral classic method of solution. In particular the use of the infinite integral of Carson, which is one general relationship resulting from Fourier analysis. This is the most fruitful process, being especially valuable in checking a result obtained by doubtful methods.

3. Processes involving the contour integral classical method of solution which is closely allied with (2). This leads to the expansion theorem as one result which is convenient for use in cases where the number of roots is small, and occasionally for an infinite number of roots.

4. The transformation of the operator into a form, the parts of which have been evaluated. Such, for example, is the scheme of splitting into partial fractions.

5. The interpretation of the product of two operators by the use of Borel's theorem.

6. The transformation of the operand, as by a shifting process we shall study.

7. The expansion of the operator in series, and interpretation term by term. This scheme was much used by Heaviside. It has the great advantage that usually it can be readily carried out to obtain a result, and the corresponding disadvantage that this result is often in a form that is not convenient for study or computation. It also involves the study of asymptotic series, which is a most difficult matter.

8. Machine processes of solving differential equations, which will be mentioned.

9. As a last resort, by the experimental process of actually setting up the physical network, or its analogue, and measuring it experimentally.

There may be other possible methods of evaluating operational expressions, but the above list includes the principal processes. We shall now consider these various methods more in detail.

3. **Algebraic Transformation of Operators.** In the first place, we know that we are justified in **transforming** $\frac{1}{Z(p)}$ **algebraically** when it is applied to the unit function.

To every operational equation of the form:

$$A(t) = \frac{1}{Z(p)} 1 \tag{203}$$

there corresponds, by the integral theorem:

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda. \tag{115}$$

Now we may juggle the left-hand side of this integral algebraically in any way we please provided the changed expression is actually algebraically equivalent to $\frac{1}{aZ(a)}$. We may develop in series, split into partial fractions or the like, all without altering the validity of the integral theorem. The same $A(t)$ corresponds to each of the altered functions we obtain on the left. Hence the operational equation, which

expresses the same relationship, may be similarly transformed; and to each transformed expression will correspond the valid infinite integral showing that we have not altered the value of $A(t)$.

4. **The Integrating Operation.** The **simplest operational expression** is

$$\frac{1}{p^m} 1$$

where m is a positive integer. To this corresponds the integral:

$$\frac{1}{a^{m+1}} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda. \tag{204}$$

We can find tables[1] of such infinite integrals and identify this one, thereby arriving at the form of $A(\lambda)$. Or we may suspect that the usual scheme for integrations holds, and that hence

$$\frac{1}{p^m} 1 = \frac{t^m}{m!}. \tag{205}$$

This may be checked readily by inserting

$$A(\lambda) = \frac{\lambda^m}{m!} \tag{206}$$

into the integral equation and performing the indicated integration. This is done by successive integration by parts until a term is arrived at which does not vanish at the limits. Since the integral equation thus reduces to an identity, the operational formula (205) is established.

The right-hand side of our expression is also discontinuous: that is, it has the value zero when $t < 0$, and $\frac{t^m}{m!}$ when $t > 0$. We can then write more carefully,

$$\frac{1}{p^m} 1 = \frac{t^m}{m!} 1 \tag{207}$$

[1] Bierens de Haan, Nouvelles Tables d'Intégrales Définies, Engels, Leiden, 1867.

to indicate this fact. Usually, however, we shall omit the final *1* except where confusion might otherwise result.

Notice that the propriety of identifying p^{-1} with $\int_0^t dt$ has been established, but only when applied to the arguments *1*, and $t^m\,1$. In general we shall find that this interpretation of p^{-1} is valid whenever the argument is zero for $t < 0$.

If we wish to evaluate $p^m t^n\,1$, we may replace the operand $t^n\,1$ by its operational expression from (205), and write

$$p^m t^n\,1 = n!\,p^m p^{-n}\,1 = n!\,p^{m-n}\,1 \qquad (208)$$
$$= \frac{n!}{p^{n-m}}\,1.$$

This is now in the form of (205), so

$$p^m(t^n\,1) = \frac{n!\,t^{n-m}}{(n-m)!}\,1 \qquad (209)$$

provided m and n are integral and both n and $(n - m) > 0$.

5. **The Exponential and the Sinusoid.** For the simple inductive circuit,

$$(R + Lp)\,i = e \qquad (210)$$

and the operational expression is thus of the form:

$$\frac{1}{p+b}\,1$$

where in this case b is R/L.

Since the current which flows in such a circuit when unit e.m.f. is applied is

$$A(t) = \frac{1}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) \qquad (211)$$

a comparison establishes the formula:

$$\frac{1}{p+b}\,1 = \frac{1}{b}(1 - \epsilon^{-bt}) \qquad (212)$$

where b is a positive real.

Again we may compare with the integral theorem. In fact, the same result has already been derived by the expansion theorem. Now if

$$(p + b)^{-1}$$

is expanded by the **binomial theorem,**

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots \quad (213)$$

or, by actually performing the division, the result is

$$(p + b)^{-1}1 = (p^{-1} - p^{-2}b + p^{-3}b^2 - \cdots)1. \quad (214)$$

We are justified in applying this term by term, for this corresponds simply to splitting A into parts in the infinite integral, and $p^{-n}1$ has been evaluated. So

$$(p + b)^{-1}1 = t - \frac{t^2b}{2!} + \frac{t^3b^2}{3!} - \cdots \quad (215)$$

and this series is recognized as the expansion of

$$\frac{1}{b}(1 - \epsilon^{-bt}).$$

Thus the operational formula has been established by a series expansion.

Now, however, expand the operator in the opposite order:

$$(b + p)^{-1}1 = (b^{-1} - b^{-2}p + b^{-3}p^2 - \cdots)1. \quad (216)$$

Here we are stopped for the present because we do not know how to interpret the terms p^n1 where n is a positive integer, and more especially we do not know how to treat an infinite series of such terms. Comparing results, we see that we must have

$$\left(\frac{p}{b} - \frac{p^2}{b^2} + \cdots\right)1 = \epsilon^{-bt} \quad (217)$$

to get a correct result.

Heaviside often encountered such series. Sometimes he threw them away, arguing that $p1$ is certainly zero when

$t > 0$; at other times he saved them and equated them to an exponential, as we have, with the statement that "there are zeros and zeros." We need to go into this matter carefully, and shall return to it later.[1]

Just as

$$\frac{p}{p+b}1 = \epsilon^{-bt}1 \tag{218}$$

where b is a positive real, so it would seem reasonable to write

$$\frac{p}{p-b}1$$

as $\epsilon^{bt}1$. This, however, is not nearly so simple a matter as it looks, as will be shown in Chapter XIII. We can use this if we recognize exactly what it means — it evidently corresponds to a circuit with negative resistance — but for the present we shall avoid impedance functions having a root with a positive real portion. There is no difficulty, though, with operators involving roots which are pure imaginaries.

Now, if we substitute $j\omega$ for b,

$$\left.\begin{aligned} \frac{1}{p+j\omega}1 &= \frac{1}{j\omega}(1-\epsilon^{-j\omega t}) \\ \frac{1}{p-j\omega}1 &= -\frac{1}{j\omega}(1-\epsilon^{j\omega t}). \end{aligned}\right\} \tag{219}$$

Adding these, there results

$$\frac{2p}{p^2+\omega^2}1 = \frac{1}{j\omega}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) \tag{220}$$

or

$$\frac{p\omega}{p^2+\omega^2}1 = \frac{1}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) = \sin \omega t. \tag{221}$$

Subtracting,

$$\frac{\omega^2}{p^2+\omega^2}1 = 1 - \cos \omega t. \tag{222}$$

[1] Chapter XII.

Notice that, if p is identified with the derivative, and if $p1 = 0$, then p can be applied to both sides of this last equation and the previous equation obtained. It is not apparent here why such a procedure will lead to a correct result, and in fact an identical procedure in other cases can lead to contradictions. A term cannot be thus arbitrarily discarded, even a term such as $p1$. We shall return to this matter presently.

If p is applied to formula (221),

$$\frac{p^2}{p^2 + \omega^2} 1 = \cos \omega t. \tag{223}$$

This also requires justification, for on the right-hand side there is a derivative of a discontinuous function. At present, however, justification can be obtained by use of the integral theorem. The infinite integral corresponding to the last expression is

$$\frac{a}{(a^2 + \omega^2)} = \int_0^\infty \epsilon^{-a\lambda} \cos \omega\lambda \, d\lambda \tag{224}$$

which can be checked from a table of infinite integrals, or by working it out.

6. **The Importance of Retaining $p1$.** The operational formula:

$$\frac{p}{p + b} 1 = \epsilon^{-bt} \tag{225}$$

corresponds to the integral equation:

$$\frac{1}{a + b} = \int_0^\infty \epsilon^{-a\lambda} \epsilon^{-b\lambda} \, d\lambda. \tag{226}$$

Now, if both sides of equation (225) are operated on with $(p + b)$,

$$(p + b) \epsilon^{-bt} 1 = p 1 \tag{227}$$

where the first 1 is inserted to remind us that we deal with the discontinuous exponential.

This may be written

$$p\epsilon^{-bt} 1 = -b\epsilon^{-bt} 1 + p1. \tag{228}$$

Notice that this does not accord with the formula which holds for the continuous exponential:

$$f(p)\ \epsilon^{bt} = f(b)\ \epsilon^{bt}. \tag{229}$$

In this case, is the $p1$ an essential part of formula (228) or may it be discarded? Let us fall back on physical interpretations. The expression:

$$p\epsilon^{-bt}1$$

is the operational expression for the response when an e.m.f. ϵ^{-bt} is suddenly applied at $t = 0$ to a pure condenser, without resistance present. Before considering this exponential e.m.f., let us consider a unit e.m.f. The response when unit e.m.f. is suddenly applied to a pure unit condenser is a pure unit impulse, that is

$$A(t) = p1. \tag{230}$$

This is consistent with the complete form of the infinite integral equation (115a).[1] It was shown there that the non-impulsive portion of the indicial admittance of a pure condenser is simply $\overline{A}(t) = 0$. The additional term to represent the impulse may be obtained by considering the charge instead of the current. When the switch is closed connecting a unit pure condenser with a source of unit e.m.f. we know that a unit charge is placed on the condenser instantly, that is,

$$q = 1. \tag{231}$$

The time function which represents the current for this case is obtained by applying p to this equation. Thus the complete form of the indicial admittance of a unit pure condenser is

$$A(t) = p1. \tag{232}$$

The response for $\epsilon^{-bt}1$ is obtained by using the complete superposition theorem. The voltage suddenly applied is unity at $t = 0$, so the impulse is again $p1$. The integral

[1] Page 83.

in the theorem drops out, and the last term, $CE'(t)$, becomes $-b\epsilon^{-bt}1$. So the correct formula is

$$p\epsilon^{-bt}1 = -b\epsilon^{-bt}1 + p1 \tag{233}$$

and the $p1$ is an **essential portion** of this equation and has a real physical meaning. If the $p1$ were left off, we should soon run into inconsistencies and contradictions.[1]

We can now see more clearly why applying p to

$$\frac{\omega^2}{p^2+\omega^2}1 = 1 - \cos\omega t \tag{234}$$

gives

$$\frac{p\omega}{p^2+\omega^2}1 = \sin\omega t. \tag{235}$$

When p is applied to the discontinuous cosine,

$$p\cos\omega t\,1 = -\omega\sin\omega t\,1 + p1. \tag{236}$$

This follows from the fact that $\cos(0) = 1$. We can also see it readily if we insert the operational form for the cosine, and transform:

$$\begin{aligned} p\cos\omega t\,1 &= p\left(\frac{p^2}{p^2+\omega^2}\right)1 = \left(p - \omega\frac{p\omega}{p^2+\omega^2}\right)1 \\ &= -\omega\sin\omega t\,1 + p\,1. \end{aligned} \tag{237}$$

Thus if p is applied to the right-hand side of (234), two terms of the form $p1$ are obtained; and these terms cancel.

7. **Partial Fractions and Inversion of Operational Formulas.** Operators can sometimes be **split into partial fractions** for interpretation. Thus,

$$\frac{p^2}{p^2+\omega^2}1 = \frac{p}{2(p-j\omega)}1 + \frac{p}{2(p+j\omega)}1. \tag{238}$$

Each of these is recognizable as the exponential; so

$$\frac{p^2}{p^2+\omega^2}1 = \frac{1}{2}(\epsilon^{j\omega t} + \epsilon^{-j\omega t}) = \cos\omega t. \tag{239}$$

[1] See V. Bush, Note on Operational Calculus, Jour. Math. and Phys., Vol. III, No. 2, March, 1924.

Similarly,

$$\frac{p\omega}{p^2+\omega^2}\,1 = \frac{p}{2j(p-j\omega)}\,1 - \frac{p}{2j(p+j\omega)}\,1$$
$$= \frac{1}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t})$$
$$= \sin \omega t. \tag{240}$$

Operational expressions can also be **inverted** to arrive at formulas with operands other than the unit function. Thus,

$$\frac{p^2+\omega^2}{p\omega}\sin \omega t\,1 = 1 \tag{241}$$

or

$$\left(\frac{p}{\omega}+\frac{\omega}{p}\right)\sin \omega t\,1 = \cos \omega t\,1 + \omega\int_0^t \sin \omega t\,1\,dt = 1. \tag{242}$$

The justification for integrating in this manner appears below.

Evidently a large number of relationships can readily be obtained in this manner.

8. **Product of Operators. Borel's Transformation.** The **product of two operators** can be transformed by means of the superposition theorem or by a theorem due to Borel,[1] which is usually more convenient.

Note at the outset that

$$\frac{1}{Z_1(p)}\cdot\frac{1}{Z_2(p)}\,1 \neq \frac{1}{Z_1(p)}\,1\cdot\frac{1}{Z_2(p)}\,1 \tag{243}$$

just as

$$\log \sin x$$

is not equal to $\log x$ multiplied by $\sin x$. However, it is possible to proceed as follows: Suppose the results of the individual operators

$$\left.\begin{aligned}\frac{1}{Z_1(p)}\,1 &= A_1(t)\\ \frac{1}{Z_2(p)}\,1 &= A_2(t)\end{aligned}\right\} \tag{244}$$

[1] Discussed by Carson, Electric Circuit Theory and the Operational Calculus, *loc. cit.*, p. 41.

are known. Then

$$\frac{1}{Z_1(p)}\left(\frac{1}{Z_2(p)}\,1\right) = \frac{1}{Z_1(p)}\,A_2(t)\,1 \tag{245}$$

may be regarded as the effect of suddenly applying an electromotive force $A_2(t)$ to a network, the impedance of which is $Z_1(p)$. The result of this is given by the superposition theorem, as

$$\begin{aligned} A(t) &= \frac{1}{Z_1(p)} \cdot \frac{1}{Z_2(p)}\,1 \\ &= \frac{d}{dt}\int_0^t A_1(t-\lambda)\,A_2(\lambda)\,d\lambda \end{aligned} \tag{246}$$

or in the other order,

$$A(t) = \frac{d}{dt}\int_0^t A_1(\lambda)\,A_2(t-\lambda)\,d\lambda. \tag{247}$$

Any of the other forms of the superposition theorem may be used; as, for example

$$A(t) = A_1(t)\,A_2(0) + \int_0^t A_1(t-\lambda)\,A_2'(\lambda)\,d\lambda. \tag{248}$$

Thus, suppose we wish to evaluate

$$\frac{p}{(p+b)^2}\,1$$

where as usual b is considered as a positive real constant. This can be written

$$\frac{1}{p+b} \cdot \frac{p}{p+b}\,1$$

and we know that

$$\frac{1}{p+b}\,1 = \frac{1}{b}\,(1-\epsilon^{-bt})$$

and

$$\frac{p}{p+b}\,1 = \epsilon^{-bt}. \tag{249}$$

Inserting these in the last form of the superposition theorem,

$$\frac{1}{b}(1 - \epsilon^{-bt}) - \int_0^t (1 - \epsilon^{-b(t-\lambda)})\, \epsilon^{-b\lambda}\, d\lambda =$$

$$\frac{1}{b}(1 - \epsilon^{-bt}) + \frac{1}{b}\epsilon^{-bt} + t\,\epsilon^{-bt} - \frac{1}{b} = t\,\epsilon^{-bt}. \qquad (250)$$

So

$$\frac{p}{(p+b)^2}\,1 = t\,\epsilon^{-bt}. \qquad (251)$$

This process can be continued, giving

$$\frac{p}{(p+b)^n}\,1 = \frac{t^{n-1}}{(n-1)!}\,\epsilon^{-bt}. \qquad (252)$$

If one of the individual responses is impulsive, the full form of superposition theorem must be used. Thus, for

$$\frac{p^2}{p+b}\,1 = p\,\frac{p}{p+b}\,1 \qquad (253)$$

the surge is $p\,1$, the integral portion of the extension theorem vanishes, and the added term is $-b\epsilon^{-bt}$, so

$$\frac{p^2}{p+b}\,1 = -b\epsilon^{-bt}\,1 + p\,1. \qquad (254)$$

In fact, if this be written

$$\left(p - \frac{pb}{p+b}\right)1$$

the result is immediately evident.

When we derived the superposition theorem for an impulsive network, we assumed that $A(t)$ was known. Now we are in a position to find it. If the network is impulsive the numerator of the fraction of the operator of the network is of a higher power in p than the denominator. We have just dealt with a fraction of this sort.

Consider the network of Fig. 25. The driving-point indicial admittance is readily evaluated:

$$A(t) = \frac{C_1 p(RC_2 p + 1)}{R(C_2 + C_1)p + 1} 1. \tag{255}$$

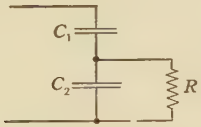

FIG. 25. Simple Impulsive Network.

This can be written

$$A(t) = \frac{C_1 C_2}{C_1 + C_2} p\, 1 + \frac{C_1^2}{C_1 + C_2} \frac{p}{R(C_1 + C_2)\, p + 1} 1. \tag{256}$$

Equation (256) includes both portions — the impulsive part, with the equivalent capacitance

$$\overline{C} = \frac{C_1 C_2}{C_1 + C_2} \tag{257}$$

and the non-impulsive part, which is evidently exponential.

There is **a theorem due to Borel** which is convenient in evaluating an operator made up of the product of two factors. This is as follows:

If

$$\left.\begin{aligned} F\,(a) &= \int_0^\infty \epsilon^{-a\lambda} f\,(\lambda)\, d\lambda \\ F_1(a) &= \int_0^\infty \epsilon^{-a\lambda} f_1(\lambda)\, d\lambda \\ F_2(a) &= \int_0^\infty \epsilon^{-a\lambda} f_2(\lambda)\, d\lambda \end{aligned}\right\} \tag{258}$$

and if

$$F(a) = F_1(a)\, F_2(a) \tag{259}$$

then

$$f(t) = \int_0^t f_1(\lambda) f_2(t - \lambda)\, d\lambda$$

$$= \int_0^t f_2(\lambda) f_1(t - \lambda)\, d\lambda. \tag{260}$$

Expressed in our usual terminology, this may be written in different form, using the integral theorem:

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda. \tag{115}$$

If

$$\frac{1}{Z_1(p)} 1 = A_1(t)$$

and

$$\frac{1}{Z_2(p)} 1 = A_2(t) \tag{261}$$

and

$$\frac{1}{Z(p)} = \frac{1}{p} \frac{1}{Z_1(p)} \frac{1}{Z_2(p)}$$

then

$$A(t) = \frac{1}{p} \frac{1}{Z_1(p)} \frac{1}{Z_2(p)} 1$$

$$= \int_0^t A_1(\lambda)\, A_2(t - \lambda)\, d\lambda$$

$$= \int_0^t A_1(t - \lambda)\, A_2(\lambda)\, d\lambda. \tag{262}$$

Notice that there are now no derivatives involved in the expression, which sometimes saves confusion, and that the operator to be evaluated is actually split into three factors, of which one is $1/p$. In using Borel's theorem we assume that neither A_1 nor A_2 is impulsive. To illustrate this theorem, let us evaluate again

$$\frac{p}{(p + b)^2} 1$$

by splitting it this time into

$$\frac{1}{p} \frac{p}{(p + b)} \frac{p}{(p + b)} 1.$$

In this manner,

$$A(t) = \frac{p}{(p+b)^2}\,1 = \int^t \epsilon^{-b\lambda}\,\epsilon^{-b(t-\lambda)}\,d\lambda$$
$$= t\epsilon^{-bt}. \qquad (263)$$

Notice that this is simpler than the use of the superposition theorem.

9. **Interpretation of p^{-1}.** Let us apply Borel's theorem to the case where

$$\left.\begin{aligned} \frac{1}{Z_1(p)}\,1 &= A_1(t) \\ \frac{1}{Z_2(p)}\,1 &= A_2(t) = 1. \end{aligned}\right\} \qquad (264)$$

Then

$$A(t) = \frac{1}{p}\,A_1(t) = \int_0^t A_1(\lambda)\,d\lambda = \int_0^t A_1(t)\,dt. \qquad (265)$$

This proves that the application of an operator $1/p$ to a function is equivalent to integrating the function between limits zero and t, provided the function is non-impulsive. Our example of the pure condenser shows how the impulsive response is treated. Another limitation is that $A(t) = 0$ when $t < 0$. This appears because the integral theorem, and hence the form of Borel's theorem obtained by combination with it, has this limitation. We cannot use either with those operators which we shall encounter below which can yield a function $A(t)$ which is not zero for negative time. Notice that we might have written just as well,

$$p^{-1}A_1(t) = \int_{-\infty}^t A_1(t)\,dt \qquad (266)$$

under the conditions governing the form of $A(t)$. In fact, this latter expression appears to apply with unusual operators when the other does not.

Similarly the application of p to a function $f(t)$ which is zero when $t < 0$ takes the usual derivative $f'(t)$ of the func-

tion, unless $f(0) \neq 0$, when there should be appended a term $f(0)\, p\, 1$, so that

$$pf(t)\, 1 = f'(t)\, 1 + f(0)\, p\, 1. \tag{267}$$

This follows immediately from our discussion of the pure condenser.

In making operational transformations the following caution should be observed. If

$$\frac{1}{Z_1(p)}\, 1 = f_1(t) \tag{268}$$

then

$$\begin{aligned} \frac{1}{Z_2(p)} f_1(t)\, 1 &= \frac{1}{Z_2(p)}\, \frac{1}{Z_1(p)}\, 1 \\ &= \frac{1}{Z_1(p)}\, \frac{1}{Z_2(p)}\, 1 \end{aligned} \tag{269}$$

and these operators can be combined as desired. But if the expression is

$$\frac{1}{Z(p)} F_1(t)\, F_2(t)\, 1$$

we cannot convert either $F_1(t)$ or $F_2(t)$ into operational form, unless both can be converted together. For example, consider

$$(p + b)\, t\epsilon^{-bt}\, 1.$$

This can be written

$$(p + b)\, \frac{p}{(p + b)^2}\, 1 = \frac{p}{p + b}\, 1 = \epsilon^{-bt}\, 1 \tag{270}$$

but it cannot be expressed as

$$(p + b)\, \frac{1}{p}\, \frac{p}{p + b}\, 1.$$

10. **Shifting Formulas.** There are certain **shifting** formulas, originally used by Heaviside, which are very convenient in transforming operational expressions by changing

the operand. Using the integral theorem:

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda \tag{115}$$

and substituting $a + b$ for a,

$$\frac{1}{a\left[\frac{a+b}{a} Z(a+b)\right]} = \int_0^\infty \epsilon^{-a\lambda} [\epsilon^{-b\lambda} A(\lambda)]\, d\lambda. \tag{271}$$

Now, just as in the first equation the function $A(t)$ corresponds to an operation $1/Z(p)$:

$$A(t) = \frac{1}{Z(p)} 1 \tag{203}$$

so in the second equation, the function

$$\epsilon^{-bt} A(t)$$

corresponds to an operation:

$$\frac{p}{p+b} \frac{1}{Z(p+b)} 1.$$

Therefore,

$$\frac{p}{p+b} \frac{1}{Z(p+b)} 1 = \epsilon^{-bt} A(t) \tag{272}$$

which shows the transformation that must be made in the operational equivalent of a time function when we multiply that function by a damping factor. In the actual evaluation of operators, however, it is the converse of this which is the more useful, as will appear later.

Variations of the above transformation may be derived which are helpful in certain cases. For instance, combining equations (272) and (203) gives

$$\epsilon^{-bt} \frac{1}{Z(p)} 1 = \frac{p}{p+b} \frac{1}{Z(p+b)} 1 \tag{273}$$

or, since

$$\frac{p}{p+b} 1 = \epsilon^{-bt} \tag{218}$$

we obtain

$$\epsilon^{-bt}\frac{1}{Z(p)}1 = \frac{1}{Z(p+b)}\epsilon^{-bt}1 \tag{274}$$

or

$$\frac{1}{Z(p)}1 = \epsilon^{bt}\frac{1}{Z(p+b)}\epsilon^{-bt}1 \tag{275}$$

or

$$\frac{1}{Z(p)}1 = \epsilon^{bt}\frac{1}{Z(p+b)}\frac{p}{p+b}1. \tag{276}$$

If this last form is reversed to read

$$\epsilon^{bt}\frac{p}{p+b}\frac{1}{Z(p+b)}1 = \frac{1}{Z(p)}1 \tag{277}$$

and we consider

$$\frac{p}{p+b}\frac{1}{Z(p+b)}1 = \frac{1}{Z_1(p)}1 \tag{278}$$

then

$$\epsilon^{bt}\frac{1}{Z_1(p)}1 = \frac{p}{p-b}\frac{1}{Z_1(p-b)}1 \tag{279}$$

or, dropping subscripts and transposing the exponential:

$$\frac{1}{Z(p)}1 = \epsilon^{-bt}\frac{p}{p-b}\frac{1}{Z(p-b)}1 \tag{280}$$

which can also be written

$$\frac{1}{Z(p)}1 = \epsilon^{-bt}\frac{1}{Z(p-b)}\frac{p}{p-b}1. \tag{281}$$

This transformation can be used provided it does not introduce operators involving positive roots; or even if it does, provided it be then interpreted correctly throughout. It is not justifiable to replace the last term of (281) by $\epsilon^{at}1$, however, without such a study, as will be shown in Chapter XIII, where the question of prospective operators is considered in detail.

We can use this process to derive the following exceedingly useful shifting formula by performing the above step on

$$\frac{1}{Z(p)}\,\frac{p}{p+b}\,1 = \frac{1}{Z(p)}\,\epsilon^{-bt}\,1. \tag{282}$$

The result is

$$\frac{1}{Z(p)}\,\epsilon^{-bt}\,1 = \epsilon^{-bt}\,\frac{1}{Z(p-b)}\,1 \tag{283}$$

which is subject to the same reservations as (280).
Note that in all these expressions operators are assumed to be non-impulsive.

The above forms are all convenient. Formula (280) enables us, when we have an operational expression, one factor of the result of which we suspect to be exponential, to write down that part immediately and make the corresponding changes in the operator. Thus, if we are evaluating

$$\frac{p\omega}{(p+\alpha)^2+\omega^2}\,1$$

we may suspect the presence of a factor $\epsilon^{-\alpha t}$, and using $b = \alpha$, write

$$\begin{aligned}\frac{p\omega}{(p+\alpha)^2+\omega^2}\,1 &= \epsilon^{-\alpha t}\,\frac{p}{p-\alpha}\,\frac{(p-\alpha)\,\omega}{(p^2+\omega^2)}\,1 \\ &= \epsilon^{-\alpha t}\,\frac{p\omega}{p^2+\omega^2}\,1 \end{aligned} \tag{284}$$

and this is recognized as

$$\frac{p\omega}{(p+\alpha)^2+\omega^2}\,1 = \epsilon^{-\alpha t}\sin\omega t. \tag{285}$$

It is sometimes convenient to have this in the form:

$$\frac{p}{ap^2+bp+c}\,1 = \frac{1}{a\sqrt{\dfrac{c}{a}-\dfrac{b^2}{4a^2}}}\,\epsilon^{-\frac{bt}{2a}}\sin\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}\,t. \tag{286}$$

Similarly

$$\frac{p^2 + \alpha p}{(p + \alpha)^2 + \omega^2} 1 = \epsilon^{-\alpha t} \cos \omega t. \tag{287}$$

Formulas for shifting other functions may also be developed. Thus, if in place of b in (273), we use first $+j\omega$ and then $-j\omega$ and add, we can form

$$\cos \omega t \frac{1}{Z(p)} 1 = \frac{p}{2}\left[\frac{1}{(p - j\omega) Z(p - j\omega)} + \frac{1}{(p + j\omega) Z(p + j\omega)}\right] 1 \tag{288}$$

and similarly

$$\sin \omega t \frac{1}{Z(p)} 1 = \frac{p}{2j}\left[\frac{1}{(p - j\omega) Z(p - j\omega)} - \frac{1}{(p + j\omega) Z(p + j\omega)}\right] 1. \tag{289}$$

Thus, for example, if we wish to form the operational expression for a damped cosine,

$$\begin{aligned} \epsilon^{-\alpha t} \cos \omega t 1 &= \cos \omega t \frac{p}{p + \alpha} 1 \\ &= \frac{p}{2}\left\{\frac{1}{p + \alpha - j\omega} + \frac{1}{p + \alpha + j\omega}\right\} 1 = \frac{p^2 + \alpha p}{(p + \alpha)^2 + \omega^2} 1 \end{aligned} \tag{290}$$

or if we wish an operational formula for $\cos^2 \omega t$,

$$\begin{aligned} \cos^2 \omega t 1 &= \cos \omega t \frac{p^2}{p^2 + \omega^2} 1 \\ &= \frac{p}{2}\left\{\frac{p - j\omega}{(p - j\omega)^2 + \omega^2} + \frac{p + j\omega}{(p + j\omega)^2 + \omega^2}\right\} 1 = \frac{p^2 + 2\omega^2}{p^2 + 4\omega^2} 1. \end{aligned} \tag{291}$$

We can also derive a general shifting formula for the exponential:

$$f(p) \epsilon^{-bt} e(t) 1 = \epsilon^{-bt} f(p - b) e(t) 1 \tag{292}$$

or in another form,

$$\frac{1}{Z(p)} \epsilon^{-bt} f(t) 1 = \epsilon^{-bt} \frac{1}{Z(p - b)} f(t) 1. \tag{293}$$

It is assumed that the operational form of the operand of the left-hand side of this expression can be written

$$\epsilon^{-bt} f(t)\, 1 = \frac{1}{F(p)}\, 1 \tag{294}$$

so that we have to transform $\frac{1}{Z(p)} \frac{1}{F(p)} 1$. By applying (280),

$$\frac{1}{Z(p)} \frac{1}{F(p)} 1 = \epsilon^{-bt} \frac{p}{p-b} \frac{1}{Z(p-b)} \frac{1}{F(p-b)} 1. \tag{295}$$

But, applying (281) to

$$\frac{1}{F(p)} 1 = \epsilon^{-bt} f(t)\, 1 \tag{296}$$

gives

$$\epsilon^{-bt} \frac{1}{F(p-b)} \frac{p}{p-b} 1 = \epsilon^{-bt} f(t)\, 1 \tag{297}$$

or

$$\frac{p}{p-b} \frac{1}{F(p-b)} 1 = f(t)\, 1 \tag{298}$$

and substituting this into (295):

$$\frac{1}{Z(p)} \frac{1}{F(p)} 1 = \epsilon^{-bt} \frac{1}{Z(p-b)} f(t)\, 1 \tag{299}$$

which proves (293). This is of course subject to the same restrictions regarding non-impulsive operators and the occurrence of positive roots as are the simpler forms from which it is derived.

11. **Transfer Formulas.** The above shifting formulas are so called because they enable us to shift a portion of an operand from one side of an operator to the other.

There is another sort of shifting or transferring process, and, to distinguish, we shall call the operators we are now to consider **transfer operators.** A transfer operator applied to a function changes the argument of the function by a constant. Consider the expression:

$$\epsilon^{-hp} f(t).$$

Expand the operator, and this becomes

$$\left(1 - hp + \frac{h^2p^2}{2!} - \cdots\right)f(t)$$

and applying it term by term:

$$f(t) - hf'(t) + \frac{h^2}{2!}f''(t) - \cdots.$$

But this last expression is simply the Taylor expansion for

$$f(t - h).$$

Thus

$$\epsilon^{-hp}f(t) = f(t - h) \tag{300}$$

and ϵ^{-hp} is a transfer operator which changes the argument of the time function to which it is applied by an amount h. This operational form of Taylor's expansion was first used by Lagrange. This transfer operator, applied to a time function, shifts the function along the axis, or introduces a delayed time. Similarly

$$\epsilon^{+hp}f(t) = f(t + h) \tag{301}$$

introduces an advanced time and shifts the function to the left along the time axis by an amount h.

The expression can be written in terms of other variables. For example,

$$\epsilon^{h\frac{d}{dx}}f(x) = f(x + h) \tag{302}$$

and $\epsilon^{h\frac{d}{dx}}$ is a transfer operator which shifts $f(x)$ a distance h along the x-axis.

So far this treatment considers only functions which have all derivatives so that they may properly be expanded in Taylor's series. We can get the same expressions for our special discontinuous functions. Thus, if

$$i(t)\,1 = \frac{1}{Z(p)}1 \tag{303}$$

we shall derive below,

$$i(t-h)\,1_h = \epsilon^{-hp}\frac{1}{Z(p)}\,1 \tag{304}$$

or

$$\epsilon^{-hp}\,i(t)\,1 = i(t-h)\,1_h. \tag{305}$$

In this expression 1_h denotes the discontinuous function which is zero when $t < h$ and unity when $t > h$.

Equation (303) corresponds to the integral:

$$\frac{1}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda}\,i(\lambda)\,d\lambda.$$

Multiply by ϵ^{-ha}:

$$\frac{\epsilon^{-ha}}{aZ(a)} = \int_0^\infty \epsilon^{-a(\lambda+h)}\,i(\lambda)\,d\lambda. \tag{306}$$

Now change the variable of this integral by replacing λ by $\lambda - h$, changing the lower limit to correspond.

$$\frac{\epsilon^{-ha}}{aZ(a)} = \int_{+h}^\infty \epsilon^{-a\lambda}\,i(\lambda-h)\,d\lambda. \tag{307}$$

Now, since $i(\lambda)$ is zero when $\lambda < 0$, $i(\lambda - h)$ is zero when $\lambda < h$, and the lower limit can be changed to zero without altering the expression. So

$$\frac{\epsilon^{-ha}}{aZ(a)} = \int_0^\infty \epsilon^{-a\lambda}\,i(\lambda-h)\,1_h\,d\lambda \tag{308}$$

which establishes

$$i(t-h)\,1_h = \epsilon^{-hp}\frac{1}{Z(p)}\,1. \tag{309}$$

If $\dfrac{1}{Z(p)}$ is unity, there results the simple expression:

$$\epsilon^{-hp}\,1 = 1_h. \tag{310}$$

We should be inclined to write immediately:

$$\epsilon^{hp}\,i(t)\,1 = i(t+h)\,1_{-h}. \tag{311}$$

Notice, however, that this introduces an important peculiarity, for an operator applied to the function $i(t)1$ which is zero when $t < 0$, yields a result which is zero when $t < -h$ and which has finite values for $-h < t < 0$. This is also evident in

$$\epsilon^{hp} 1 = 1_{-h}. \tag{312}$$

Thus no argument in regard to these expressions can be based on the integral equation. Consider

$$\epsilon^{hp} \frac{1}{p} 1.$$

In accordance with the above, this could be written

$$\epsilon^{hp} t\, 1 = (t + h)1_{-h} \tag{313}$$

which seems entirely reasonable. Now if the order of operations is reversed,

$$\frac{1}{p} \epsilon^{hp} 1 = \frac{1}{p} 1_{-h} \tag{314}$$

and it is evident that in order to arrive at the previous expression it is necessary to interpret p^{-1} as $\int_{-\infty}^{t} dt$, and not as $\int_{0}^{t} dt$. This is not surprising, for this latter interpretation does not hold for functions that are not zero for $t < 0$.

Thus if we use the integral from $(-\infty)$ in our interpretations we at least have a consistent system, although there is a gap in our argument concerning which we shall comment later on.[1]

Functions of x, when discontinuous, can be treated similarly to the above discontinuous functions of time. In particular, the operator

$$\epsilon^{-vt\frac{d}{dx}}$$

[1] H. Jefferys, Operational Methods in Mathematical Physics, Cambridge University Press, 1927.

applied to a function of x produces

$$\epsilon^{-vt\frac{d}{dx}} f(x) = f(x - vt). \tag{315}$$

If t is the time, and v a velocity, the last expression represents a curve or wave moving along the axis of x at a velocity v in the positive x direction. Similarly $\epsilon^{+vt\frac{d}{dx}}$ sets a curve in motion at constant velocity in the negative x direction. If we abbreviate $\frac{d}{dx}$ by Δ, this is written

$$\epsilon^{+vt\Delta} f(x) = f(x + vt). \tag{316}$$

These transfer operators are very convenient. Suppose, for example, in solving a wave problem, we have arrived at an operational expression for our result, containing operators of both kinds — time derivatives and space derivatives. It is of the form:

$$i(t,x) = F(p,\Delta)\, 1 \tag{317}$$

and we suspect the result describes a wave running in the positive x direction at velocity v. Then if we transform thus:

$$i(t,x) = \epsilon^{-vt\Delta} [\epsilon^{vt\Delta} F(p,\Delta)]\, 1 \tag{318}$$

the quantity in brackets should represent the wave standing still, and hence should become a simpler expression.

Transfer operators thus enable functions to be moved about in time or position at our convenience, while they are still in operational form. The use of special operators enables us to

(*a*) Remove damping factors.

(*b*) Retard or delay the appearance of an effect after the cause.

(*c*) Translate space curves to the right or left.

(*d*) Convert space curves into traveling waves.

(*e*) Make traveling waves stand still. This is one of the most powerful features of the Operational Calculus. By

means of such processes we can readily examine a traveling wave, making it stand still in space, and stripped of its attenuation, disclose its distortion with the lapse of time.

12. **Illustrative Example.** We shall give one simple illustration of the direct operational solution of a lumped circuit — that of the series R, L, and C circuit with unit applied e.m.f.

The operational equation is

$$A(t) = \frac{1}{R + Lp + \frac{1}{Cp}} 1 \tag{319}$$

and if we use as abbreviations:

$$\left.\begin{aligned} \alpha &= \frac{R}{2L} \\ \omega &= \sqrt{\frac{1}{LC} - \alpha^2} \end{aligned}\right\} \tag{320}$$

this becomes[1]

$$A(t) = \frac{1}{L\omega} \frac{p\omega}{(p+\alpha)^2 + \omega^2} 1 \tag{321}$$

which is recognized to be

$$A(t) = \frac{1}{L\omega} \epsilon^{-\alpha t} \sin \omega t. \tag{322}$$

If $\alpha > \frac{1}{\sqrt{LC}}$ we shall have a pure imaginary for ω, say $\omega = j\omega_1$. The same operational processes hold, but the final result can be changed in form.

$$\begin{aligned} A(t) &= \frac{1}{jL\omega_1} \epsilon^{-\alpha t} \sin j\omega_1 t \\ &= \frac{1}{L\omega_1} \epsilon^{-\alpha t} \sinh \omega_1 t. \end{aligned} \tag{323}$$

[1] For the interpretation given, $\frac{1}{LC} > \alpha^2$.

Once the operational formulas are available, the solution of problems becomes a matter of algebra only.

13. **Operational Treatment of Impulses.** There are some exceedingly interesting features of the operational treatment of impulses. It was by reasoning in terms of these that Heaviside arrived at his infinite integral theorem,[1] and we shall review some of his argument briefly.

We have seen that $p\,1$ represents a unit impulse, that is a function of t which is zero everywhere except at the origin where it is infinite. Moreover we have concluded from physical reasoning in terms of the charge placed on a unit condenser that the integral of $p\,1$ is unity. To say that the area under the function $p\,1$ is unity is of course only a short way of expressing an idea that really requires a more complete statement in order to be exact. We can state perfectly properly that the limit of the area under the time function,

$$\frac{p}{Rp+1}1$$

approaches unity as R approaches zero. In fact the area under this finite time function is always unity, whatever the value of R, so that the process of passing to the limit is here exceedingly simple. Of course the total charge that passes to a unit condenser when it is connected to a unit e.m.f. is unity, whether or not there is a resistance in the circuit. This is of course simply avoiding difficulty of statement by placing a small resistance in series with the unit condenser. If we realize that we actually have in mind this limiting process we can properly say that $p\,1$ has unit area.

Now

$$\epsilon^{-xp}p\,1 \tag{324}$$

[1] Electromagnetic Theory, Vol. III, p. 237. Heaviside's notation is at this point so different from his usual scheme that the theorem is not readily apparent until it is translated into the notation with which we are more familiar.

evidently represents a similar impulse of unit area placed at the point on the time axis where $t = x$. Similarly,

$$\epsilon^{-t\Delta}\Delta\, 1 \tag{325}$$

represents a unit impulse which is a function of x, and which is placed on the x-axis at the point $x = t$. These two expressions are entirely equivalent. This can be shown by examining the operational expression:

$$\frac{p\Delta}{p+\Delta}\, 1$$

where for the first time we have an operator involving the derivatives with respect to both t and x.
If this is written

$$\Delta\left(\frac{p}{p+\Delta}\right) 1$$

and the fraction treated as though Δ were a constant, it becomes

$$\Delta\epsilon^{-t\Delta}\, 1$$

which is equivalent to (325) above. In the other order,

$$p\left(\frac{\Delta}{\Delta+p}\right) 1$$

it becomes by the same reasoning,

$$p\epsilon^{-xp}\, 1$$

which is equivalent to (324).

Now suppose some function of t, say $f(t)$, be multiplied by one of these impulses and integrated:

$$\int_0^\infty f(t)(\epsilon^{-xp}\, p\, 1)\, dt.$$

The product is zero at all times except when $t = x$. Then $f(t)$ has the value $f(x)$ and the integral is simply the integral

of the impulse multiplied by this value. So

$$f(x) = \int_0^\infty f(t)(\epsilon^{-xp}\, p\, 1)\, dt. \tag{326}$$

Our transferred impulse is thus a means for picking off one particular value of a function. In just the same way,

$$f(t) = \int_0^\infty f(x)(\epsilon^{-t\Delta}\, \Delta\, 1)\, dx. \tag{327}$$

Let us transform this last equation by replacing the expression for the impulse we have used by the other which we have seen is identical with it. Also, in place of $f(t)$ let us insert its operational equivalent $\frac{1}{Z(p)}\, 1$. We then have

$$\frac{1}{Z(p)}\, 1 = \int_0^\infty f(x)(\epsilon^{-xp}\, p\, 1)\, dx. \tag{328}$$

Since the integration is against x, the p that occurs can be treated exactly as though it were a constant; and in fact it can be replaced by a constant, say a. If this is done, the *1* which appears on both sides may be dropped, giving

$$\frac{1}{Z(a)} = a \int_0^\infty \epsilon^{-ax} f(x)\, dx \tag{329}$$

which is the integral theorem as previously expressed.

As we have noted, Heaviside did not rely on this theorem to establish operational formulas; but rather employed it to evaluate infinite integrals when the result of the corresponding operational expression was known. It is of use either way. Since there are other perfectly valid ways of substantiating operational formulas besides the integral theorem, for example the expansion theorem or series developments following Volterra or Wiener, the use that Heaviside made may become the more valuable process as operational schemes are further developed. It is sometimes an exceedingly difficult job to evaluate an infinite integral otherwise.

Heaviside in fact gives some sixty-five illustrative examples of the evaluation of definite integrals by this sort of operational treatment. We need review only one to show the idea involved, for our subject is the application of operational methods to circuit analysis rather than to the solution of various types of mathematical problems.

The operational formula,

$$\epsilon^{-\frac{h}{p}} 1 = J_0 (2\sqrt{ht}) \tag{330}$$

can be established readily, for if the operator is expanded in a power series:

$$\epsilon^{-\frac{h}{p}} 1 = \left(1 - \frac{h}{p} + \frac{h^2}{2!\,p^2} - \cdots\right) 1 \tag{331}$$

and applied term by term,

$$\epsilon^{-\frac{h}{p}} 1 = \left(1 - ht + \frac{h^2t^2}{2!^2} - \frac{h^3t^3}{3!^2} + \cdots\right) \tag{332}$$

which is the series for the Bessel function we have written This process is entirely sound, as will be shown in Chapter XII. So operational formula (330) may be regarded as established.

Suppose now the infinite integral:

$$\int_0^\infty J_0 (2\sqrt{xt})\, dx$$

is encountered. Replace the Bessel function in this integral by its operational equivalent:

$$\int_0^\infty \left(\epsilon^{-\frac{x}{p}} 1\right) dx.$$

Integration with p as a constant gives

$$\int_0^\infty J_0 (2\sqrt{xt})\, dx = p\, 1. \tag{333}$$

The value of the integral is therefore zero, except for the special case when $t = 0$, in which case it is infinite. This

remarkable result is obtained with surprising ease, although some steps have been passed over rather rapidly — steps that should have been examined closely if the operational scheme were to be used continually in evaluating integrals.

In carrying on the processes of this section, Heaviside was really arguing in terms of what are now known as Stieltjes integrals.[1] It is probable that his work was entirely independent of that of Stieltjes; and though Heaviside evolved the treatment in a manner entirely lacking in mathematical completeness and rigor, he undoubtedly fully appreciated the philosophy of integrals of this sort by which impulses can be treated so conveniently.

There is evidently a very close connection between the Stieltjes integral and the operational processes of Heaviside — a connection which promises to be useful when it is further developed.

Heaviside's scheme of using the integral theorem, which we shall obtain as a Fourier transform in Chapter X, is extremely suggestive, and it would be well worth while to examine more carefully the conditions surrounding some of his steps. The scheme of replacing a function by its operational expression in order to evaluate an integral is undoubtedly connected with the question of reversing the order of integration in a Fourier integral. Of the many examples which he gives in Vol. III, there are several which furnish much food for thought.

[1] T. J. Stieltjes, Recherches sur les Fractions Continues Ann. Fac. sc. Toulouse (1) **8**, 1894, p. 1; Oeuvres Completes de Thomas Jan Stieltjes Groningen, Tome II, 1918, p. 402.

Hildebrandt, On Integrals Related to and Extensions of the Lebesgue Integrals, Bull. Am. Math. Soc., **24**, 1917–18, pp. 113, 177.

J. Hadamard, Course d'Analyse, Paris, 1927, p. 97.

PROBLEMS ON CHAPTER VIII

Prob. 1–8. Evaluate $\frac{p}{ap^2 + bp + c}\,1$.

Prob. 2–8. Evaluate $\frac{p^2}{(p + a)(p + b)}\,1$ in the following ways:

a. By splitting into partial fractions.

b. As a product of factors; using the superposition theorem. Prove this result by use of the infinite integral theorem.

Prob. 3–8. Find the operational expression for $\epsilon^{-at}(1 - at)\,1$:

a. By treating it as the sum of two recognizable terms.

b. By use of the infinite integral theorem.

Prob. 4–8. Find the operational expression for $t\epsilon^{-bt}\,1$ by expanding the time function in series, converting to operational form term by term, and then summing this operational series.

Prob. 5–8. An operational expression

$$\frac{1}{Z(p)}\,1$$

can be factored to give

$$\frac{1}{p^2} \cdot \frac{1}{Z_1(p)} \cdot \frac{1}{Z_2(p)} \cdot \frac{1}{Z_3(p)}\,1$$

where the results of $\frac{1}{Z_1(p)}\,1$, $\frac{1}{Z_2(p)}\,1$, and $\frac{1}{Z_3(p)}\,1$ are known. How can Borel's theorem be used to evaluate $\frac{1}{Z(p)}\,1$?

Prob. 6–8. Evaluate $\frac{p}{p^3 + bp^2 + \omega^2 p + b\omega^2}\,1$ by use of Borel's theorem.

Prob. 7–8. Find the operational expression corresponding to

$$t\epsilon^{-at} \sinh kt\,1.$$

Prob. 8–8. Evaluate the following operational expression and prove the result by means of the infinite integral equation

$$\frac{1}{(p + b)^2}\,1.$$

Prob. 9–8. Evaluate and plot $\frac{p\omega}{p^2 + \omega^2}\left(1 + \epsilon^{-\frac{\pi p}{\omega}}\right)1$.

Prob. 10–8. Review and discuss several of Heaviside's examples in Electromagnetic Theory Vol. III, p. 237.

Prob. 11–8. In the circuit of Fig. 25 write the operational formula for the transfer indicial admittance between the terminals shown and a

point in the resistance, and show that no impulsive term appears. Formulate by the superposition theorem the current that will flow at this point when a sinusoidal voltage is impressed on the network at the instant when the voltage is a maximum.

Prob. 12–8. In Prob. 7–2 set up and evaluate the operational expression for the motion of the pendulum.

Prob. 13–8. Evaluate the operational expressions

$$(a).\quad \frac{1}{(p+\alpha)(p+\beta)(p+\gamma)}\,1$$

$$(b).\quad \frac{p+b_1}{(p+\alpha)(p+\beta)(p+\gamma)}\,1$$

$$(c).\quad \frac{p^2+b_1p+b_2}{(p+\alpha)(p+\beta)(p+\gamma)}\,1$$

$$(d).\quad \frac{p^3+b_1p^2+b_2p+b_3}{(p+\alpha)(p+\beta)(p+\gamma)}\,1$$

Prob. 14–8. Evaluate $\frac{p}{\sqrt{p+a}}\epsilon^{-at}\sqrt{t}\,1$. Note that $p^{-\frac{1}{2}}1 = 2\sqrt{\frac{t}{\pi}}$.

Prob. 15–8. To what operational formula does the following infinite integral correspond?

$$\sqrt{\frac{b}{\pi}}\int_0^\infty \epsilon^{-a\lambda}\frac{\epsilon^{-\frac{b}{\lambda}}}{\lambda\sqrt{\lambda}}\,d\lambda = \epsilon^{-2\sqrt{ba}}.$$

Prob. 16–8. Given the operational equation $\frac{p}{(p+a)^2}\,1 = t\epsilon^{-at}\,1$ operate on this with ϵ^{-bp} and evaluate the result.

Prob. 17–8. Evaluate the operational expression

$$\left(\frac{kp-1}{kp+1}\right)^2 1.$$

Prob. 18–8. Discuss the difference in meaning of $\frac{f^2(t)}{Z^2(p)}$ and $\left[\frac{f(t)}{Z(p)}\right]^2$.

CHAPTER IX

FUNCTIONS OF A COMPLEX VARIABLE

It is hardly possible to properly grasp the Heaviside operational method without some knowledge of the theory of functions of a complex variable. In fact, there is so close an interdependence that any attempt to extend the method or to use it in unusual cases will be decidedly hampered unless this theory is freely available.

On the other hand, even a rudimentary knowledge of the theory is sufficient for some purposes. An engineer with such a knowledge can safely proceed to an extent that involves a basic understanding rather than a simple parrot-like use of formulas, and can depend upon the theory of functions to raise a danger signal in doubtful cases.

To the electrical engineer complex function theory comes rather naturally, the plane vector scheme of solving alternating-current circuits in the steady state having made him familiar with complex quantities and their rules of combination. He has also used circle diagrams. He has even formulated functions of complex variables and used diagrams for their representation.[1] But line integrals in the complex plane, and the study of discontinuities, are usually completely left out of his equipment unless he has had more than an undergraduate course.

We cannot present here even a brief superficial review of any considerable part of the subject. It will only be possible to interpret some of the language of the subject so that explanations can be made in terms of it.

2. **Impedance as a Function of a Complex Variable.** Suppose we consider the operator

$$f(p)1 = \frac{1}{Z(p)}\,1. \tag{334}$$

[1] Kennelly's "Chart Atlas of Complex Hyperbolic and Circular Functions" is an excellent example of what is meant.

In the examination by means of Heaviside's expansion theorem it appears that the roots of the equation

$$Z(\lambda) = 0 \tag{335}$$

play an important part. These roots may be complex, but for a dissipative system they always have a real portion which is negative except under certain peculiar circumstances which will be discussed in a subsequent chapter. Thus, if these roots are plotted in the complex plane, they will in general lie on the left-hand side of the axis of imaginaries.

It will be shown later that the position and nature of these points of discontinuity, where $f(\lambda)$ becomes infinite, determine the nature of the operator with which we have to deal, and hence the process which may be safely employed in evaluating it.

3. **Line Integrals in the Complex Plane.** The integral

$$\int_a^b f(x)\, dx$$

may be regarded as a line integral taken along the axis of reals. It defines an operation which may be roughly described as follows: Divide the interval ab into segments Δx, multiply the length of each by the value of the function at any point x in that segment, add, and take the limit of the sum as Δx approaches zero.

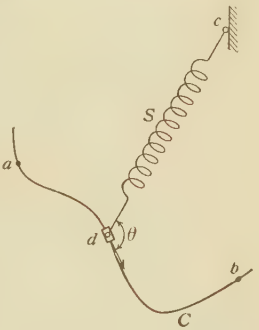

FIG. 26. Line Integral of a Force.

Similarly a line integral may be taken along other real paths. For example, suppose in Fig. 26 there is a spring S fastened at a point c and attached to a moving member at d. We desire the value of the work done in moving d along the path C. Form the function $f(s)$, which expresses the component of the force of the spring along the direction of the

path in terms of the distance s traveled along that path, and take the line integral:

$$\int_C f(s)\, ds$$

along the path from a to b. From the conservation of energy, the value of this line integral is independent of the path chosen in this case, and depends simply upon the position of the two termini, a and b. This is a line integral in a real plane, obtained from a two-dimensional problem.

A line integral in a complex plane is similarly formed.[1] The meaning of

$$\int_C f(\lambda)\, d\lambda$$

between two points, a and b, in the complex plane is as follows: The path is divided into segments $\Delta\lambda$. Each length of segment is multiplied by the value of the function at any point in that segment. The sum is taken, and then finally the limit of this sum as $\Delta\lambda$ approaches zero. In this procedure λ is of the form $x + jy$, and hence $\Delta\lambda$ is also complex.

There are much more precise definitions of the line integral than these, but for the functions usually met in circuit problems these will suffice.

There are many important theorems in regard to line integrals in a complex plane. We shall comment briefly on some of the more important.

[1] As general references on functions of a complex variable, consult J. Pierpont, Functions of a Complex Variable, Ginn & Co., 1914.

E. J. Townsend, Functions of a Complex Variable, Henry Holt & Co., 1915.

E. Goursat, Cours d'Analyse Mathématique, Tome II, Gauthier, Paris, 1905; Translation by E. R. Hedrick, Ginn & Co., 1904.

H. Durége, Elemente der Theorie der Funktionen einer Komplexen Veränderlichen Grösse, Teubner, Leipzig, 1906. Elements of the Theory of Functions of a Complex Variable, Philadelphia, 1896.

W. F. Osgood, Lehrbuch der Funktionentheorie, Teubner, Leipzig, 1923.

The value of a **line integral in the complex plane** is independent of the path taken; that is, it is the same for two different paths, provided these paths and the area enclosed by them contain no point at which the function being integrated becomes infinite. We shall consider for the present only single-valued functions. To illustrate:

$$f_1(\lambda) = \frac{\lambda}{\lambda + 1} \tag{336}$$

is single-valued; that is, to every value of λ in the complex plane there corresponds only one value of f. But

$$f_2(\lambda) = \sqrt{\lambda} \tag{337}$$

is double-valued, for to each value of λ in general correspond two values of f. When $\lambda = 1, f$ is ± 1. When $\lambda = j$,

$$f = \pm\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right).$$

The function $f_1(\lambda)$ becomes infinite at the point $\lambda = -1$. The line integral (Fig. 27) of $f_1(\lambda)$ along Path A is the same as along B, but the value is different along C. **It follows from the above that the integral around a closed path is zero provided there is no point of discontinuity on or within the path of integration.** Such a path can be split into two parts, along which the values of the integral are equal and opposite.

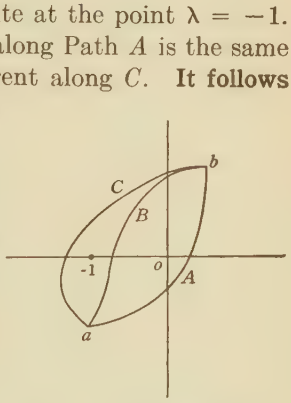

FIG. 27. Dependence of Line Integral on Path.

This last theorem can be restated. The path of integration may be stretched or contracted in any way we please, provided in so doing we do not pass over a point of discontinuity.

The value of a line integral about a closed path in a given direction is equal to the sum of the integrals about small closed paths in the same direction, each inclosing one point

of discontinuity in the original area. This follows from contracting the path until parts coincide, and then canceling out these parts. The positive direction around the path is conventionally taken as counter-clockwise. Thus in Fig. 28 if the crosses indicate points of discontinuity of $f(\lambda)$, the integral $\int f(\lambda)\, d\lambda$ around C is equal to the sum of the integrals around C_1, C_2, and C_3.

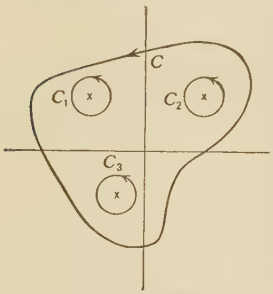

FIG. 28. Deformation of a Path of Integration.

4. **Residues and their Evaluation.** There are two kinds of points of discontinuity at which functions become infinite — **poles** and **essential singularities.** At a pole a function becomes infinite to a finite order, while at an essential singularity it becomes infinite to an infinite order. If λ_1 is a point at which $f(\lambda)$ becomes infinite,

$$\lim_{\lambda \to \lambda_1} f(\lambda) = \infty . \tag{338}$$

If λ_1 is a pole,

$$\lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n f(\lambda) \neq \infty \tag{339}$$

provided n is chosen large enough, but finite. If λ_1 is an essential singularity,

$$\lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n f(\lambda) \tag{340}$$

does not exist no matter what finite value is taken for n.

The order of a pole is the value of n required to remove the discontinuity. Thus $\lambda/(\lambda + 1)$ has a first-order pole at $\lambda_1 = -1$. But $\epsilon^{\frac{1}{\lambda}}$ has an essential singularity at $\lambda = 0$.

The value of the line integral of a function around a path which encloses a pole is equal to $2\pi j$ times the residue $\mathcal{R}$

of the function at the pole. If the pole is of the first order, the residue

$$\mathfrak{R} = \lim_{\lambda \to \lambda_1} (\lambda - \lambda_1) f(\lambda). \tag{341}$$

Thus the residue at $\lambda_1 = -1$ of $\lambda/(\lambda + 1)$ is

$$\lim_{\lambda \to -1} (\lambda + 1) \frac{\lambda}{\lambda + 1} = -1 \tag{342}$$

and the value of the line integral of $\lambda/(\lambda + 1)$ around a path which encloses this point is $-2\pi j$.

Similarly the line integral of $1/\lambda$ about the origin is $2\pi j$, for $\lim_{\lambda \to 0} \lambda \left(\frac{1}{\lambda}\right) = 1$. Let this simple case be examined in detail by actually performing the integration.

To evaluate

$$\int_C \frac{d\lambda}{\lambda}$$

change to polar coördinates to simplify the integration. Let

$$\begin{aligned} \lambda &= \rho(\cos\theta + j \sin\theta) \\ &= \rho\epsilon^{j\theta} = \rho\angle\theta. \end{aligned} \tag{343}$$

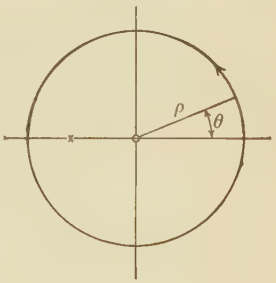

FIG. 29.

Take as a path a circle about the origin, Fig. 29. Along this path ρ is constant, and

$$\begin{aligned} d\lambda &= \rho(-\sin\theta + j\cos\theta)\, d\theta \\ &= j\rho\epsilon^{j\theta}\, d\theta. \end{aligned} \tag{344}$$

The positive direction is counter-clockwise. So

$$\int_C \frac{d\lambda}{\lambda} = \int_0^{2\pi} \frac{j\rho\epsilon^{j\theta}\, d\theta}{\rho\epsilon^{j\theta}} = j\int_0^{2\pi} d\theta = 2\pi j. \tag{345}$$

On the other hand, let the integral of this same function be evaluated about a circle of radius r and center at $\lambda = a$

(Fig. 30). Here ρ is not constant, and it will hence be best to change variable to simplify the expression. If

$$\lambda = a + re^{j\phi} \tag{346}$$

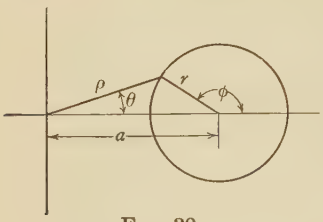

FIG. 30.

and as r is constant,

$$d\lambda = jre^{j\phi}\, d\phi$$

and the integral becomes

$$\int_0^{2\pi} \frac{jre^{j\phi}}{a + re^{j\phi}}\, d\phi.$$

The numerator is seen to be the exact derivative of the denominator, and the integral evaluates into the logarithm of the complex radius vector, which gives, when putting in the limits, 0 when $r < a$ and $2\pi j$ when $r > a$.

However, the evaluation will be made in terms of real integrals. Write the integral as

$$jr\int_0^{2\pi} \frac{\cos\phi + j\sin\phi}{(a + r\cos\phi) + jr\sin\phi}\, d\phi$$

and transform into

$$jr\int_0^{2\pi} \frac{a\cos\phi + r}{a^2 + r^2 + 2ar\cos\phi}\, d\phi - r\int_0^{2\pi} \frac{a\sin\phi}{a^2 + r^2 + 2ar\cos\phi}\, d\phi. \tag{347}$$

The second integral is zero from symmetry, as will be seen on examining Fig. 31, which is a plot of the various factors and the integrand.

The first integral becomes[1]

$$\begin{aligned} &j\frac{\phi}{2}\Big]_0^{2\pi} + \frac{j}{2}(r^2 - a^2)\int_0^{2\pi} \frac{d\phi}{a^2 + r^2 + 2ar\cos\phi} \\ &\quad = j\pi + j(r^2 - a^2)\int_0^{\pi} \frac{d\phi}{a^2 + r^2 + 2ar\cos\phi}. \end{aligned} \tag{348}$$

This follows from symmetry, as seen in Fig. 32, and the change is made to avoid ambiguity in the integration. This

[1] Peirce, A Short Table of Integrals, *loc. cit.*, #304, 300.

can now be integrated, giving

$$j\pi \pm j \sin^{-1}\left[\frac{2\,ar + (a^2 + r^2)\cos\phi}{(a^2 + r^2) + 2\,ar\cos\phi}\right]_0^{\pi}$$
$$= j\pi \pm j\,[\sin^{-1}(-1) - \sin^{-1}(1)] = j\pi \pm j\pi. \quad (349)$$

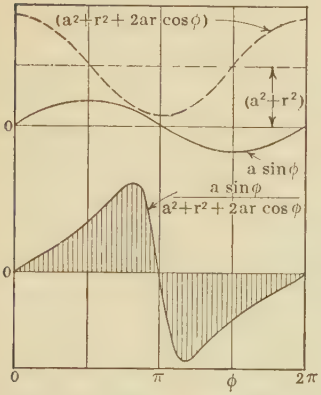

FIG. 31. Plot of Second Integrand of Equation (347) and its Components.

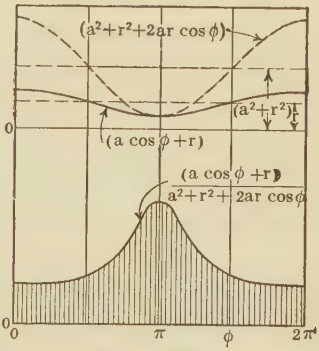

FIG. 32. Plot of First Integrand of (347) and Components for $r > a$.

The curves are again consulted to remove ambiguity of sign. Fig. 32 is drawn for $r > a$ and Fig. 33, for $r < a$. Evidently the + sign belongs with $r > a$, and we obtain $2\pi j$ for the integral whose path encloses the pole, and zero otherwise. Incidentally this example illustrates the relative simplicity that occurs in evaluating integrals in the complex plane by means of residues as compared to the actual process of detailed integration. It is this simplification which many times makes an argument in

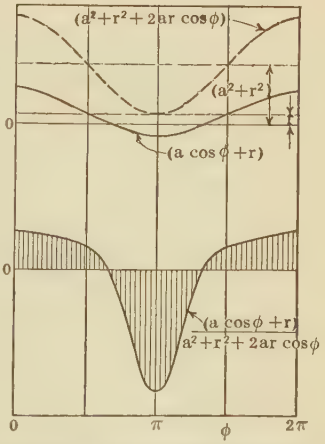

FIG. 33. Plot of First Integrand of (347) and Components for $r < a$.

terms of integrals in the complex plane preferable to one in terms of ordinary integrals.

5. **Derivation of Expansion Theorem by Contour Integral.** We are now in a position to derive the Heaviside expansion theorem by means of a **contour integral.**[1] The expression:

$$\frac{1}{2\pi j}\int \frac{\epsilon^{\lambda t}}{\lambda}\, d\lambda \tag{350}$$

taken along the path shown in Fig. 34, represents our familiar unit function. It will be noted that the integrand has a pole of unit order at the origin, and that this is avoided by choosing a path that cuts around this point by a small half circle to the right, which is a common artifice. To show that this expression represents *1* we shall show that it is zero for $t < 0$, and unity for $t > 0$.

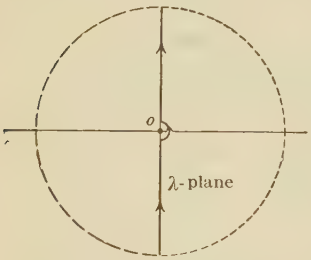

FIG. 34. Path of Integration of (350).

When $t < 0$, the integrand becomes exponentially zero for large values of λ to the right of the axis of imaginaries. Hence the dotted half circle, of very large radius and in the right-half plane, can be added without altering the value of the integral. Remote from the axis of imaginaries this is self-evident. For parts near this axis some examination of this point may be desirable, and a brief discussion of the justification for closing the path will be included below. The path is now closed. It encloses no points at which the integrand is infinite. Hence the integral is zero.

[1] Bromwich, and Wagner, *loc. cit.*, utilized this method. Wagner's proof is reviewed by Bush, Proc. I.R.E., 1917, p. 377. The work of Bromwich forms the basis for a tract by Harold Jefferys, Operational Methods in Mathematical Physics, *loc. cit.*, in which appear also several new and interesting analyses of wave and diffusion problems.

When $t > 0$ a similar half circle can be added to the left, as shown by dashes. This closed path surrounds the origin, which is a pole. The value of the integral is hence

$$2\pi j\left[\lim_{\lambda \to 0} \lambda \frac{\epsilon^{\lambda t}}{\lambda}\right] = 2\pi j.$$

Expression (350) is thus equal to unity for $t > 0$, and is a formal expression which is identically equal to the unit function. Moreover it is expressed as the limit of a sum of terms of the form $\epsilon^{\lambda t}$ which are exponential functions of the time t.

In the above the question of whether it is legitimate to add to the path of integration, when $t > 0$, a half circle of large radius to the left of the axis of imaginaries was passed over rather lightly. The integral along such a half circle will evidently be zero unless an appreciable contribution is obtained from the portion of the path immediately adjacent to the axis of imaginaries, for remote from this axis the integrand approaches zero exponentially as the radius of the circle is increased. We shall therefore examine the value of the integral taken over a short path of length b projecting out from the axis of imaginaries toward the left. If $\lambda = x + jy$, it will be sufficient to regard y as constant over this short path, and integrate with respect to x. This gives

$$\int_0^{-b} \frac{\epsilon^{xt}\epsilon^{jyt}}{x + jy}\, dx$$

and if the value of this integral approaches zero as y is increased, it will be legitimate to add the half circle. This integral is rather hard to handle, but inspection shows that its value is certainly less than that of

$$\int_0^{-b} \frac{\epsilon^{jyt}}{x + jy}\, dx$$

so that if this latter approaches zero the proposition will have been demonstrated. Integration with y as a constant

gives

$$\epsilon^{jyt} \log (x + jy)\Big]_0^{-b} = \epsilon^{jyt} \log\left(j\frac{b}{y} + 1\right). \qquad (351)$$

Now as y is increased this quantity in parenthesis approaches one, and the logarithm of one is zero. The factor ϵ^{jyt} is never greater than one. So the value of the integral approaches zero as y is increased, and the addition of a half circle was proper. The propriety of adding a half circle to the right when $t < 0$ can be treated in identical manner.

When a continuous exponential voltage $\epsilon^{\lambda t}$ is applied to a circuit of impedance function $Z(\lambda)$, the current that flows is $\epsilon^{\lambda t}/Z(\lambda)$. Since a continuous exponential term of this sort exists for all values of t, this is the only response, and no transient term is needed.

The effect of applying unit e.m.f. to a circuit may be found by considering that each exponential term produces its own current term, and hence

$$A(t) = \frac{1}{2\pi j}\int \frac{\epsilon^{\lambda t}}{\lambda Z(\lambda)}\, d\lambda \qquad (352)$$

taken along the same path shown by the heavy line in Fig. 34.

Now, with a dissipative network, all the roots of $Z(\lambda) = 0$ will in general lie to the left of the axis of imaginaries; that is, they will have negative real portions. Moreover, for a non-impulsive lumped circuit they will all be finite. Such points are poles of our integrand, shown by crosses in Fig. 35, where for simplicity only three roots have been shown. The path can be deformed as before, and for the same reason, provided $Z(\lambda)$ is an algebraic fraction with a denominator of no higher power than the numerator, (which is the case for

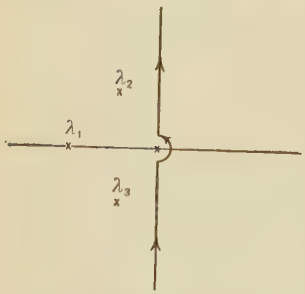

FIG. 35. Location of Poles of Integrand of (352).

lumped non-impulsive networks), for the effect of introducing $Z(\lambda)$ is then to decrease, or at least not to increase by more than a constant factor the integrand on a circle of large radius. For $t < 0$ the path again encloses no poles, and the current is hence zero, as it should be. For $t > 0$ the completed path now encloses the pole at the origin and all the poles corresponding to roots of Z. The path can be deformed into a small circle about each of these (Fig. 36).

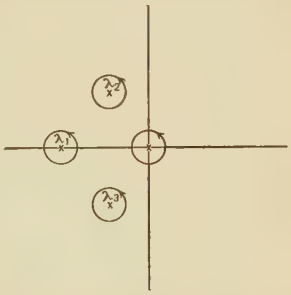

FIG. 36. Deformed Path of Integration (352) for $t > 0$.

We are now ready to evaluate. The value of the expression will be the sum of the residues of the integrand at the poles. The residue at the origin is

$$\lim_{\lambda \to 0} \lambda \frac{\epsilon^{\lambda t}}{\lambda Z(\lambda)} = \frac{1}{Z(0)}. \tag{353}$$

For the pole λ_k the residue is

$$\lim_{\lambda \to \lambda_k} (\lambda - \lambda_k) \frac{\epsilon^{\lambda t}}{\lambda Z(\lambda)}.$$

To evaluate this, write

$$Z(\lambda) = \frac{D(\lambda)}{M(\lambda)}$$

where, as in Chapter VII, D and M have been reduced to polynomials.

Now

$$D(\lambda) = b(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_k) \ldots (\lambda - \lambda_{2n}) \tag{354}$$

so

$$Z(\lambda) = \frac{b(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_k) \ldots (\lambda - \lambda_{2n})}{M(\lambda)}. \tag{355}$$

The residue is then

$$\frac{M(\lambda_k)\epsilon^{\lambda_k t}}{b\lambda_k(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)\ldots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\ldots(\lambda_k-\lambda_{2n})}.$$

The sum of the residues then gives

$$A(t) = \frac{1}{Z(0)} + \sum \frac{M(\lambda_k)\epsilon^{\lambda_k t}}{b\lambda_k(\lambda_k-\lambda_1)(\lambda_k-\lambda_2)\ldots} \qquad (356)$$

where for each root there is a term in the summation which includes the product of all factors obtained by taking the difference between that root and the others. This is one form of the expansion theorem. It can be changed, as in Chapter VII, to the form:

$$A(t) = \frac{1}{Z(0)} + \sum \frac{\epsilon^{\lambda_k t}}{\lambda_k Z'(\lambda_k)}. \qquad (357)$$

Now this derivation shows that the expansion theorem holds in this form when: (1) the roots of $D(\lambda)$ are all distinct and different from zero, giving first-order poles; (2) all roots lie to the left of the axis of imaginaries, as is the usual case with a dissipative network; (3) $Z(\lambda)$ is single-valued; (4) the network is a lumped, non-impulsive network.

These conditions are much the same as those involved in the proof of Chapter VII, where however Z was limited to be of the form of a rational algebraic fraction with numerator of higher power than the denominator, and with distinct roots different from zero. This present derivation allows the numerator and denominator to be of the same power. Physically this means that we have now demonstrated the expansion theorem for the case where there occurs an initial finite value of current at the instant the switch is closed.

When all roots are not distinct, special treatment is necessary. Wagner's form for coincident roots was given. It is readily derived by an extension of the above, but in full form is so highly complicated as to be difficult to use. Hence it was given only in simple form with undetermined constants.

The expansion theorem, or for that matter, any other operational processes discussed so far, cannot be used on networks involving negative resistances, without a special inquiry which will be instituted in Chapter XIII. The condition that $Z(\lambda)$ be single-valued will be treated in Chapter XI.

The theorem has been proved for a finite number of roots. In distributed systems the impedance functions have an infinite number of roots, and we have studied an illustration of that sort. Bromwich discussed the case of an infinite number of roots, and Fry[1] mentioned it briefly. It can be examined by a process which will merely be outlined roughly.

If $Z(\lambda)$ has an infinite number of roots, evenly spaced, or at least not concentrating on any point, the path of integration can be deformed by adding a half circle as before, of finite radius R. R is chosen of such a value that the path does not cut a pole. The path now includes a finite number of poles, and the evaluation is carried on as above. The result is an approximate expression for $A(t)$. Now R is chosen larger, to include one more pole, and a term is added to the expression. A continuation of this process makes the value of the integral on the half circle approach zero, and thus the contour integral approaches $A(t)$. If the series developed as a result converges and hence approaches a limit, it will also approach $A(t)$. This is a skeleton of the proof only. It does not appear that the theorem has ever been rigorously established for cases involving an infinite number of roots. There is in fact another matter that would need to be examined carefully in any such demonstration. In such cases the generating function is not algebraic, and the propriety of closing the path needs examination. It is certain that there are many transcendental forms of the

[1] T. C. Fry, The Solution of Circuit Problems, Phys. Rev., **14**, 1919, p. 115; The Application of Modern Theories of Integration to the Solution of Differential Equations, Annals of Math., **22**, 1921, p. 182. In these articles Fry uses the contour integral in a powerful manner to develop several processes of much interest.

generating function for which this cannot be accomplished as outlined, and no general criterion appears to have been produced as a guide. In particular, if the function is exponential it is evident that the integral along a path of large radius cannot be ignored. The expansion theorem must therefore be regarded as firmly established only for lumped non-impulsive networks. Fortunately it is rare that it is needed elsewhere, for distributed networks yield best to methods employing series or to the direct operational attack. The expansion theorem was much used on diffusion problems by Heaviside, and this has been repeated by some later workers, but without complete justification.

It might be thought, perhaps, that the use of the contour integral method, which so quickly involves us in this dilemma of the propriety of closing a path is a maneuver that is simply unnecessarily borrowing trouble. On the contrary, it brings out the actual inherent limitations of a theorem, which might possibly be overlooked when examined by other methods.

Carson developed a form of expansion theorem for use when the applied voltage is of the form $\epsilon^{j\omega t}\ 1$ instead of the unit function, and Fry proved it by use of the contour integral. Since it follows readily from the above, it is given here. The current $i(t)$ in response to $\epsilon^{j\omega t}\ 1$ is desired. This form of applied voltage may be written

$$\frac{1}{2\pi j}\int \frac{\epsilon^{j\omega t}\epsilon^{\lambda t}}{\lambda}\, d\lambda = \frac{1}{2\pi j}\int \frac{\epsilon^{(\lambda+j\omega)t}}{\lambda}\, d\lambda \qquad (358)$$

with the same path and comments as before. The response is

$$i(t) = \frac{1}{2\pi j}\int \frac{\epsilon^{(\lambda+j\omega)t}}{\lambda Z(\lambda + j\omega)}\, d\lambda. \qquad (359)$$

This has poles at the origin, and at the values of λ such that

$$Z(\lambda + j\omega) = 0, \text{ namely } \lambda = \lambda_1 - j\omega,\ \lambda_2 - j\omega \ldots .$$

The residue at the origin is

$$\lim_{\lambda \to 0} \lambda \frac{\epsilon^{(\lambda + j\omega)t}}{\lambda Z(\lambda + j\omega)} = \frac{\epsilon^{j\omega t}}{Z(j\omega)} \tag{360}$$

which is the steady-state term.

At the pole, $\lambda_k - j\omega$, the residue is

$$\lim_{\lambda \to (\lambda_k - j\omega)} \frac{(\lambda - \lambda_k + j\omega)\epsilon^{(\lambda + j\omega)t} M(\lambda + j\omega)}{\lambda(\lambda - \lambda_1 + j\omega)(\lambda - \lambda_2 + j\omega) \ . \ . \ .}$$

and

$$i(t) = \frac{\epsilon^{j\omega t}}{Z(j\omega)} + \sum \frac{M(\lambda_k)\epsilon^{\lambda_k t}}{(\lambda_k - j\omega)(\lambda_k - \lambda_1)(\lambda_k - \lambda_2) \ . \ . \ .} \tag{360a}$$

$$= \frac{\epsilon^{j\omega t}}{Z(j\omega)} + \sum \frac{\epsilon^{\lambda_k t}}{(\lambda_k - j\omega)Z'(\lambda_k)} . \tag{360b}$$

This formula is of course subject to the same restrictions as those outlined above.

A treatment of exceptional cases of this form of the expansion theorem, especially those involving resonance, is given by Gotō.[1]

PROBLEMS ON CHAPTER IX

Prob. 1–9. By means of contour integration and residue theory, find the indicial admittance of a simple R and L series circuit. Indicate the path and positive direction for each integration on a plot of the complex plane.

Prob. 2–9. By means of contour integration and residue theory, find the indicial admittance of the simple R, L, and C series circuit, assuming that the constants are such as to give the oscillatory case.

Prob. 3–9. In the derivation of the Heaviside expansion theorem by contour integration in the complex plane, in what way is each of the following limitations on the validity of the theorem introduced?

a. Restriction to dissipative networks.
b. No repeated roots.
c. No zero roots.
d. Restriction to non-impulsive networks.
e. Restriction to lumped-parameter networks.

[1] Mochinori Gotō, Extension of the Heaviside Expansion Theorem, *loc. cit.*

Prob. 4–9. Evaluate the operational expression

$$\frac{p}{p+a}\,\epsilon^{-at}\,t\,1$$

by means of contour integration.

Prob. 5–9. Evaluate $\frac{p}{(p+a)^2}\,1$ by the use of a contour integral. Avoid coincident roots by evaluating first $\frac{p}{(p+a)\,(p+a+h)}$, and then letting h approach zero. Discuss the physical meaning of this process.

CHAPTER X

THE FOURIER INTEGRAL FORMULATION

Heaviside's attitude toward formal mathematics, or at least toward some formal mathematicians, was at times rather censorious. The battle was not at all one-sided, and we can well have sympathy with both parties. It is startling to read his complaint of the "rejection of honest work," undoubtedly his own, by some "learned society" when "papers are so cheap." Yet it is not at all hard to understand the point of view of the editors or judges.

He lived in a time when mathematics was being given an ardent rigorous overhauling — which experience has proved was much needed — when loosenesses were being ironed out, and floating hypotheses nailed down. He introduced a new and radical mathematical attack on physical problems which was very powerful but also very obviously full of holes. Zealots are seldom patient, long-suffering, or charitable, whether they discuss prohibition or mathematics. Conflict was inevitable. It is only unfortunate that it resulted finally in the early loss of a great pioneer and drove him away from the completion of his idea into a rigorous entity, which possibly he might otherwise have carried through. Battles — mental battles between scientists — are delightful when both parties are logical and capable, provided no one gets hurt. This particular conflict, however, was serious in its results to society.

We need both the precursor and the rigorist in mathematics as in nearly everything else. Fortunately we do not need many of either type, for their lot is often not one of ease. Rather the ordinary mathematician or physicist or engineer needs to have many of the characteristics of each in his make-up and be able to change his mode of thought as

the chameleon changes his color to suit his surroundings. It is usually unproductive of results to set a dyed-in-the-wool accountant at inventing, and no one wants a radical in mathematics at work on the company's books.

As we study the operational method, we still need the attitude of Heaviside — the experimental, courageous, uncritical aspect which will try anything in mathematical processes once, and obtain checks later on if they can be found. But this attitude belongs only in those parts of the subject which lie on the unmapped borderland. It is inexcusable to be vague when it is reasonably possible to be sure. We can place our operational steps on firm ground much more frequently than could Heaviside a generation ago.

The real attitude of Heaviside toward formal mathematics was vastly different from what has become somewhat the popular concept. So we shall begin this chapter by quoting Heaviside.[1]

"And I must here write a caution. I may have to point out sometimes that my method leads to solutions much more simply than Fourier's method. I may therefore appear to be disparaging and endeavoring to supersede his work. But it is nothing of the sort. In a complete treatise on diffusion Fourier's and other methods would come in side by side — not as antagonists but as mutual friends helping one another. . . . No one admires Fourier more than I do. It is the only entertaining mathematical work I ever saw. Its lucidity has always been admired. But it was more than lucid. It was luminous. Its light showed a crowd of followers the way to a heap of new physical problems."

The Fourier analysis is at the basis of the operational method.

2. **Fourier Series.** We shall first review some fundamental ideas in regard to Fourier series. If $f(t)$ is a single-valued function which is so restricted that it has only a finite

[1] Electromagnetic Theory, Vol. II, p. 32.

number of discontinuities and of maxima and minima between $t = -s$ and $t = s$, and is such that

$$\int_{-s}^{s} [f(t)]^2 \, dt$$

exists, then the **Fourier Series**

$$\tfrac{1}{2} b_0 + b_1 \cos\frac{\pi t}{s} + b_2 \cos\frac{2\pi t}{s} + \cdots + a_1 \sin\frac{\pi t}{s} + a_2 \sin\frac{2\pi t}{s} + \cdots \tag{361}$$

where

$$\left.\begin{aligned} b_h &= \frac{1}{s}\int_{-s}^{s} f(t) \cos\frac{h\pi t}{s}\, dt \\ a_h &= \frac{1}{s}\int_{-s}^{s} f(t) \sin\frac{h\pi t}{s}\, dt \end{aligned}\right\} \tag{362}$$

converges to $f(t)$ for all values of t between $t = -s$ and $t = s$, excepting the values of t corresponding to the discontinuities of $f(t)$, and the values $t = -s$ and $t = s$ for the case in which $f(-s)$ is not equal to $f(s)$.

At the points of discontinuity the series is equal to the

$$\lim_{u \to 0} \tfrac{1}{2} [f(t + u) + f(t - u)]$$

provided this limit exists. This is the intermediate value of the function at the point, and is given by half the sum of the respective values which the function takes on at this point upon approaching it first from the left and then from the right.

At s and $-s$, if $f(s)$ is not equal to $f(-s)$, the value of the series is equal to the

$$\lim_{u \to 0} \tfrac{1}{2} [f(-s + u) + f(s - u)]$$

provided this limit exists.

The same values of the series are repeated in the interval s to $3s$, etc. Hence if $f(t)$ is periodic of period $2s$, and is

subject to the above restrictions, it can be represented for the entire range of t by this trigonometric series which may be written

$$f(t) = \frac{b_0}{2} + \sum_1^\infty b_h \cos\frac{h\pi t}{s} + \sum_1^\infty a_h \sin\frac{h\pi t}{s}. \qquad (363)$$

The process of finding the series to represent a given periodic function is called harmonic analysis. There are various mechanical devices for mechanically or graphically evaluating the coefficients of such a series.[1] They are all mechanisms for evaluating integrals of the type:

$$\frac{1}{s}\int_{-s}^{s} f(t) \sin\frac{h\pi t}{s}\, dt$$

which gives the coefficient for the hth sine component. The process evidently consists of multiplying the given function by a sinusoid of the proper period and obtaining the average of the product between $-s$ and s. Evidently the series can be written in the form:

$$f(t) = \tfrac{1}{2} b_0 + \sum_1^\infty g_h \sin\left(\frac{h\pi t}{s} + \theta_h\right) \qquad (364)$$

where

$$g_h = \sqrt{a_h^2 + b_h^2} \qquad (365)$$

is the amplitude of the hth harmonic, and

$$\theta_h = \tan^{-1}\frac{b_h}{a_h} \qquad (366)$$

is its phase angle. As an example, let us find the series for the function of Fig. 37, which is often encountered in wave theory. Here the coefficients of all cosine terms, and the

[1] Consult, for instance, J. Lipka or F. S. Dellenbaugh, *loc. cit.*, or E. M. Horsburgh, Napier Tercentenary Handbook, 1914.

A. Galle, Mathematische Instrumente, 1912.

F. A. Laws, Electrical Measurements, McGraw-Hill Book Co., 1917, p. 666.

constant term, are zero by symmetry. The value of a_h is, also by symmetry,

$$a_h = \frac{2}{s}\int_0^s \sin\frac{h\pi t}{s}\,dt$$
$$= \frac{2}{h\pi}(1 - \cos h\pi) \qquad (367)$$

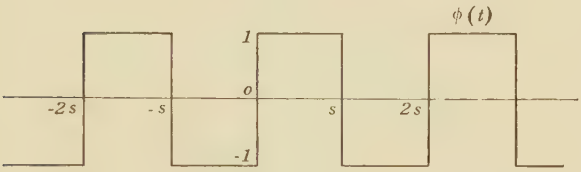

FIG. 37. The Function $\phi(t)$ of Equation (368).

and this is zero when h is even, and $4/h\pi$ when h is odd. Thus the series is[1]

$$\phi(t) = \frac{4}{\pi}\left(\sin\frac{\pi t}{s} + \tfrac{1}{3}\sin\frac{3\pi t}{s} + \cdots\right)$$
$$= \frac{4}{\pi}\sum_1^\infty \frac{1}{h}\sin\frac{h\pi t}{s} \quad h \text{ odd.} \qquad (368)$$

Only odd sine terms appear because the function is symmetrical and odd, with identical positive and negative loops. Note that at points of discontinuity, namely at $-2s$, $-s$, 0, s, $2s$, etc., the series is equal to zero, which is the intermediate value of $\phi(t)$ at these points.

3. **Operational Equivalents.** Functions which can be represented by Fourier series can also often be conveniently expressed by operational expressions. These are often compact and simple in form. The operational expression appears to be a natural way of expressing a discontinuous function.

An operational expression can be easily written for the function of the preceding section, or rather for the function

[1] Plots of the partial sums will be found in W. E. Byerly, Fourier Series and Spherical Harmonics, Ginn & Co., 1893, p. 63, or in Ingersoll and Zobel, Mathematical Theory of Heat Conduction, Ginn & Co., 1913.

which is equal to it for $t > 0$, and zero for $t < 0$; that is $\phi(t)1$. The hth term is

$$\sin\frac{h\pi t}{s}\,1 = \frac{\dfrac{h\pi p}{s}}{p^2 + \left(\dfrac{h\pi}{s}\right)^2}\,1. \tag{369}$$

Hence

$$\phi(t)1 = \frac{4}{\pi}\sum_1^\infty \frac{1}{h}\frac{\dfrac{h\pi}{sp}}{1+\left(\dfrac{h\pi}{sp}\right)^2}\,1,\ h \text{ odd}$$

$$= \frac{2}{sp}\sum_1^\infty \frac{2}{1+\left(\dfrac{h\pi}{sp}\right)^2}\,1. \tag{370}$$

Now there is a partial-fraction series development of the hyperbolic tangent of the form:[1]

$$\tanh\frac{x}{2} = \frac{2}{x}\sum_{m=1}^\infty \frac{2}{1+\left(\dfrac{m\pi}{x}\right)^2},\ m \text{ odd}. \tag{371}$$

Comparing these last two expressions,

$$\phi(t)1 = \tanh\frac{sp}{2}\,1. \tag{372}$$

[1] Partial-fraction series developments of trigonometric functions will be found in most texts on complex function theory. See for example Hurwitz-Courant, Funktionentheorie, Springer, Berlin, 1925, p. 123, where appears

$$\pi\tan\pi z = -\sum_{-\infty}^{\infty}\left(\frac{1}{z-\dfrac{2m-1}{2}} + \frac{1}{\dfrac{2m-1}{2}}\right).$$

From this, by multiplying both sides by z and making the substitution $z = \dfrac{x}{2\pi}$, and then splitting the sum into parts,

$$\frac{x}{2}\tan\frac{x}{2} = -\frac{1}{1+\dfrac{\pi}{x}} - \sum_{m=1}^\infty\left(\frac{1}{1-(2m-1)\dfrac{\pi}{x}} + \frac{1}{1+(2m+1)\dfrac{\pi}{x}}\right)$$

$$+\frac{1}{\dfrac{\pi}{x}} - \sum_{m=1}^\infty\left(\frac{1}{(2m-1)\dfrac{\pi}{x}} - \frac{1}{(2m+1)\dfrac{\pi}{x}}\right)$$

where the terms outside the summation signs come from the zero term. Now the last two terms combine to give zero, as is seen on writing out

This may be checked by means of the integral

$$\frac{1}{a}\tanh\frac{sa}{2} = \int_0^\infty \epsilon^{-a\lambda}\phi(\lambda)\,d\lambda. \tag{373}$$

Split the integral into intervals from 0 to s, s to $2s$, etc. $\phi(\lambda)$ is one in the first integral, **minus one** in the second, etc. On integrating and inserting limits, the right-hand side becomes

$$\frac{1}{a}(1 - 2\,\epsilon^{-sa} + 2\,\epsilon^{-2sa} - \cdots).$$

Express the left-hand side of the equation in exponential form:

$$\frac{1}{a}\frac{1 - \epsilon^{-sa}}{1 + \epsilon^{-sa}}$$

and an identity results when the indicated division is performed.

As a simple example of the occurrence of this operator, consider a **transmission line** of length D, with inductance l and capacitance c per unit of length, open at the distant end and devoid of resistance or leakance. The alternating-current impedance of an open line[2] of this length is

$$\sqrt{\frac{r + jl\omega}{g + jc\omega}}\coth D\sqrt{(r + jl\omega)(g + jc\omega)}.$$

---

the sum. The first two terms, similarly examined, yield

$$\frac{x}{2}\tan\frac{x}{2} = -\sum_{m=1}^{\infty}\left(\frac{2}{1 - \left(\frac{m\pi}{x}\right)^2},\ m \text{ odd.}\right)$$

Replace x by jx, and use $\tan jx = j\tanh x$,

$$\frac{x}{2}\tanh\frac{x}{2} = \sum_{1}^{\infty}\frac{2}{1 + \left(\frac{m\pi}{x}\right)^2},\ m \text{ odd.}$$

Other partial-fraction developments are similarly obtained.

See also W. B. Ford, Studies on Divergent Series and Summability, Macmillan Co., 1916, p. 32.

[2] A. E. Kennelly, Artificial Electric Lines, McGraw-Hill Book Co., 1917, p. 52.

For the case of no dissipation, the impedance operator is thus

$$\sqrt{\frac{l}{c}} \coth \frac{Dp}{v}$$

where

$$v = \frac{1}{\sqrt{lc}}. \tag{374}$$

If l is the inductance computed from flux external to the wire, v will be the velocity of light or the velocity of wave-front propagation on the line. When unit electromotive force is applied to such a line, the entering current is

$$i_0(t) = \sqrt{\frac{c}{l}} \frac{1}{\coth \frac{Dp}{v}} 1 = \sqrt{\frac{c}{l}} \tanh \frac{Dp}{v} 1. \tag{375}$$

The current is thus

$$i_0(t) = \sqrt{\frac{c}{l}} \phi(t) 1. \tag{376}$$

Now $s = \frac{2D}{v}$ in (372), and thus the period of oscillation of the fundamental is $4D/v$; or the wave length of the fundamental is four times the length of line.

4. **Limitations of the Fourier Representation.** It must not be thought, however, that the Fourier series or its operational equivalent is an expression which is identical in all its behavior with the function being represented.[1] If $f(t)$ is restricted as in Sec. 2, then the Fourier series converges to $f(t)$ for any value of t for which the function is continuous. From a physical standpoint this means that the graph of the series can be made to coincide as closely as desired with the graph of the function, except at points of discontinuity, by taking a sufficient number of terms of the series.

[1] For a discussion of this subject see H. S. Carslaw, Fourier Series and Integrals, MacMillan, London, 1921. For the function we have discussed see p. 274.

The integral of the series is, under very general conditions, the integral of the function; but the derivative of the series, term by term, does not in general converge to the derivative of the function. To be convinced of this, plot the series discussed above for several terms and add a term at a time. Note that the ripples decrease in size but do not become flattened out.

If a Fourier series is obtained as an end result in a problem, it will represent the solution as closely as need be. But if a result is obtained in a problem which expresses the charge on a condenser as a Fourier series, a valid expression for the current cannot in general be obtained by differentiating this series term by term.

5. **Fourier Integrals.** If the expressions for a_h and b_h are inserted directly into the Fourier series equation (361), and λ is used as the variable of integration to avoid confusion, then

$$f(t) = \frac{1}{2s}\int_{-s}^{s} f(\lambda)\,d\lambda + \sum_{1}^{\infty}\frac{1}{s}\sin\frac{h\pi t}{s}\int_{-s}^{s} f(\lambda)\sin\frac{h\pi\lambda}{s}\,d\lambda + \sum_{1}^{\infty}\frac{1}{s}\cos\frac{h\pi t}{s}\int_{-s}^{s} f(\lambda)\cos\frac{h\pi\lambda}{s}\,d\lambda \quad (377)$$

or

$$f(t) = \frac{1}{2s}\int_{-s}^{s} f(\lambda)\,d\lambda + \sum_{1}^{\infty}\frac{1}{s}\int_{-s}^{s} f(\lambda)\left[\sin\frac{h\pi t}{s}\sin\frac{h\pi\lambda}{s} + \cos\frac{h\pi t}{s}\cos\frac{h\pi\lambda}{s}\right]d\lambda$$

$$= \frac{1}{2s}\int_{-s}^{s} f(\lambda)\,d\lambda + \sum_{1}^{\infty}\frac{1}{s}\int_{-s}^{s} f(\lambda)\cos\frac{h\pi}{s}(\lambda - t)\,d\lambda. \quad (378)$$

Now in this series, s may be taken as large as desired. The Fourier integral is suggested but not rigorously established by the form this series appears to take as s is made infinite.[1]

[1] For a rigorous proof establishing the Fourier integral, and for fewer restrictions on the arbitrary function, see E. W. Hobson, The Theory of Functions of a Real Variable, Cambridge University Press, 1926, Vol. II.

To do this, note that $\cos(-\theta) = \cos\theta$, and write equation (378) in the form

$$f(t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\frac{\pi}{s}\int_{-s}^{s} f(\lambda)\cos\frac{h\pi}{s}(\lambda - t)\,d\lambda \qquad (379)$$

which expanded gives

$$\begin{aligned}
\cdots &+ \frac{1}{2\pi}\int_{-s}^{s} f(\lambda)\frac{\pi}{s}\cos\left\{\left(-\frac{2\pi}{s}\right)(\lambda - t)\right\}d\lambda \\
&+ \frac{1}{2\pi}\int_{-s}^{s} f(\lambda)\frac{\pi}{s}\cos\left\{\left(-\frac{\pi}{s}\right)(\lambda - t)\right\}d\lambda \\
&+ \frac{1}{2\pi}\int_{-s}^{s} f(\lambda)\frac{\pi}{s}\cos\left\{\left(\frac{0\,\pi}{s}\right)(\lambda - t)\right\}d\lambda \\
&+ \frac{1}{2\pi}\int_{-s}^{s} f(\lambda)\frac{\pi}{s}\cos\frac{\pi}{s}(\lambda - t)\,d\lambda \\
&+ \frac{1}{2\pi}\int_{-s}^{s} f(\lambda)\frac{\pi}{s}\cos\frac{2\pi}{s}(\lambda - t)\,d\lambda + \cdots \qquad (380)
\end{aligned}$$

This sum will have a limit as s approaches ∞ provided $f(t)$ is restricted as in Sec. 2, and in addition is such that

$$\lim_{s\to\infty}\int_{-s}^{s}[f(t)]^2\,dt$$

exists. The limiting value of the series will be

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}f(\lambda)\cos\xi\,(\lambda - t)\,d\lambda \qquad (381)$$

where $\Delta\xi = \frac{\pi}{s}$. This is the **Fourier Integral.**[1] This will represent any arbitrary function $f(t)$, restricted as above, at all points in the infinite interval from $-\infty$ to $+\infty$ except

[1] Electrical engineers are finding the Fourier integral of increasing usefulness. A few practical applications are given by J. R. Carson, Electric Circuit Theory and the Operational Calculus, Chapter XI. Selective Circuits and Static Interference, B.S.T.J., **4**, 1925, p. 265.

J. W. Milnor, Submarine Cable Telegraphy, Trans. A.I.E.E., **41**, 1922, p. 20.

at points of discontinuity where, as was the case for the series, it will be equal to

$$\lim_{u \to 0} \tfrac{1}{2} [f(t + u) + f(t - u)]$$

provided this limit exists.

Since $\cos \xi(\lambda - t) = \cos - \xi(\lambda - t)$, equation (381) can be written

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(\lambda) \cos \xi(t - \lambda)\, d\lambda. \qquad (382)$$

Expressing the cosine in exponential form and making the change of variable $\omega = j\xi$, the following is obtained:

$$f(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} d\omega \int_{-\infty}^{\infty} f(\lambda) \frac{\epsilon^{\omega(t-\lambda)} + \epsilon^{-\omega(t-\lambda)}}{2} d\lambda.$$

If this is split into the sum of two integrals and a change of variable made in one of them, it will be seen that they are identical, so

$$f(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} d\omega \int_{-\infty}^{\infty} f(\lambda) \epsilon^{\pm\omega(t-\lambda)}\, d\lambda. \qquad (383)$$

6. **Network Solution by a Fourier Integral.** Now suppose a voltage $e(t)$, which is subject to the same restrictions as $f(t)$ above, is to be applied to a network at time $t = 0$. This voltage can then be represented by

$$e(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} d\omega \int_{-\infty}^{\infty} \epsilon^{\omega(t-\lambda)} e(\lambda)\, d\lambda. \qquad (384)$$

Now $e(t)$ is expressed as the limit of a sum of terms of exponential form:

$$\epsilon^{\omega(t-\lambda)} \quad \text{or} \quad \epsilon^{\omega t} \epsilon^{-\omega\lambda}.$$

Suppose the operator of the network is $f(p)$. Then to any such exponential term of voltage will correspond a term of current,

$$f(\omega)\, \epsilon^{\omega(t-\lambda)}$$

and the complete expression for current will be[1]

$$i(t) = f(p)e(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} d\omega \int_{-\infty}^{\infty} f(\omega)\epsilon^{\omega(t-\lambda)}e(\lambda)\, d\lambda. \quad (385)$$

Moreover, since $e(t)$ is suddenly applied at $t = 0$, that is $e(t) = 0$ when $t < 0$, the limit can be changed with the following result:

$$i(t) = f(p)e(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} f(\omega)\, d\omega \int_{0}^{\infty} \epsilon^{\omega(t-\lambda)}e(\lambda)\, d\lambda. \quad (386)$$

Thus the response of the network to voltage $e(t)$ has been expressed entirely in terms of the operational function f by means of a Fourier integral. Any way that this can be evaluated will thus give the solution of our problem.

7. **Explicit Expressions for the Indicial Admittance.** The Fourier integral expression, as it has been used here, is limited in its application to a voltage $e(t)$ which is **of summable square,** that is

$$\int_{-\infty}^{\infty} [e(t)]^2\, dt$$

exists, so that $e(t)$ certainly approaches zero for large values of t. Wiener has an extended form of integral which is not so limited, but at this point the ordinary form will be used. Let the applied voltage then be of the form ϵ^{-at} applied at $t = 0$. The expression then becomes

$$\begin{aligned} i(t) &= f(p)\epsilon^{-at}\, 1 = f(p)\,\frac{p}{p+a}\, 1 \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} f(\omega)\, d\omega \int_{0}^{\infty} \epsilon^{\omega(t-\lambda)}\epsilon^{-a\lambda}\, d\lambda \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon^{\omega t} f(\omega)\, d\omega \int_{0}^{\infty} \epsilon^{-(a+\omega)\lambda}\, d\lambda. \end{aligned} \quad (387)$$

[1] Many of the transformations of this chapter will be found in Giorgi, Sul Calcolo delle Soluzioni Funzionali, *loc. cit.*

Performing this second integration:

$$f(p)\epsilon^{-at}\,1 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{f(\omega)}{a+\omega}\,\epsilon^{\omega t}\,d\omega. \tag{388}$$

Notice that the path of integration is along the axis of imaginaries and that the integrand is regular along this path, unless $f(\omega)$ has poles on this axis, which occurs only when dissipation is neglected and which requires special treatment.

Now let us limit ourselves for the moment to a network in which the steady-state term is zero when unit electromotive force is applied. In such a network $f(0) = 0$. Since the current in such a network ultimately dies down to zero, the response of such a network will be exactly the same when ϵ^{-at} is suddenly applied, as when unit e.m.f. is suddenly applied, provided the value of a is small enough. Therefore we have, as closely as we please, for such a network,

$$A(t) = f(p)1 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{f(\omega)}{\omega}\,\epsilon^{\omega t}\,d\omega. \tag{389}$$

This is the same expression that was utilized in Chapter IX for development into a contour integral; except, because of the assumption that $f(0) = 0$, there is now no pole at the origin and the integration can be carried out entirely along the axis of imaginaries. Notice that except for certain cases of impulsive networks the integrand is still regular along the path of integration. Moreover, if there is a steady-state term, we can treat, in place of $f(p)$, the operator:

$$F(p) = f(p) - f(0)$$

which has no such term, and can add the steady-state term independently:

$$A(t) = f(0)1 + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{f(\omega) - f(0)}{\omega}\,\epsilon^{\omega t}\,d\omega \tag{390}$$

where a 1 has been appended to the $f(0)$ as a reminder that it is to be included only when $t > 0$. The entire expression

is now zero when t is negative. Now change the variable by the substitution:

$$\omega = j\omega'$$

$$A(t) = f(0)1 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon^{j\omega' t}\frac{f(j\omega') - f(0)}{j\omega'}\, d\omega'. \quad (391)$$

We now have a real variable of integration and an ordinary integral along the axis of reals. Dropping the prime, to simplify notation,

$$A(t) = f(0)1 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon^{j\omega t}\frac{f(j\omega) - f(0)}{j\omega}\, d\omega. \quad (392)$$

This is an explicit expression for $A(t)$. It is expressed directly in terms of the alternating-current admittance $f(j\omega)$ of the network. It is not an integral equation for determining $A(t)$, but $A(t)$ directly expressed as an infinite integral. It can be used for computing $A(t)$ when the generating function $f(j\omega)$ is known, but it is more convenient to transform it somewhat for this purpose.

$f(j\omega)$ is the current which flows in the network in response to voltage $\epsilon^{j\omega t}$; that is, if the symbolic method of expressing facts regarding alternating-current circuits in the steady state is adopted, it is the response of the network to unit voltage of angular velocity ω. This admittance can be split into its power and quadrature components:

$$f(j\omega) = \frac{1}{Z(j\omega)} = P(\omega) + jQ(\omega). \quad (393)$$

Thus $P(\omega)$ is the power component of the current as a function of the frequency for unit alternating voltage at standard phase, and $Q(\omega)$ is the quadrature component. Inserting these, and writing

$$\epsilon^{j\omega t} = \cos \omega t + j \sin \omega t \quad (394)$$

we have

$$A(t) = f(0)1 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{j\omega}[\cos \omega t + j \sin \omega t]$$
$$[P(\omega) - f(0) + jQ(\omega)]\, d\omega. \quad (395)$$

Now this can be split into its real and imaginary portions:

$$A(t) = f(0)1 + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{P(\omega)-f(0)}{\omega}\sin\omega t + \frac{Q(\omega)}{\omega}\cos\omega t\right]d\omega$$
$$-\frac{j}{2\pi}\int_{-\infty}^{\infty}\left[\frac{P(\omega)-f(0)}{\omega}\cos\omega t - \frac{Q(\omega)}{\omega}\sin\omega t\right]d\omega. \tag{396}$$

The current $A(t)$ is a real. Hence this last term must vanish. It then follows that

$$\int_{-\infty}^{\infty}\frac{P(\omega)-f(0)}{\omega}\cos\omega t\,d\omega = \int_{-\infty}^{\infty}\frac{Q(\omega)}{\omega}\sin\omega t\,d\omega \tag{397}$$

so that

$$A(t) = f(0)1 - \frac{f(0)}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\omega t}{\omega}\,d\omega$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{P(\omega)}{\omega}\sin\omega t + \frac{Q(\omega)}{\omega}\cos\omega t\right)d\omega. \tag{398}$$

Now the infinite integral:

$$\int_{-\infty}^{\infty}\frac{\sin\omega t}{\omega}\,d\omega \tag{399}$$

has the value π when $t > 0$. Hence for $t > 0$,

$$A(t) = \frac{f(0)}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{P(\omega)}{\omega}\sin\omega t + \frac{Q(\omega)}{\omega}\cos\omega t\right)d\omega. \tag{400}$$

At zero frequency there is no quadrature component, so

$$Q(0) = 0 \quad \text{and} \quad f(0) = P(0).$$

So

$$A(t) = \tfrac{1}{2}P(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{P(\omega)}{\omega}\sin\omega t + \frac{Q(\omega)}{\omega}\cos\omega t\right)d\omega. \tag{401}$$

The expression (398) is valid for any value of t. Rewrite it, substituting $-t$ for t. We know that $A(-t) = 0$. Also the first term on the right disappears when t is negative. The values of the integrals involving the sine change sign when

$-t$ is substituted for t. Consequently, for $t > 0$,

$$0 = \tfrac{1}{2} P(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Q(\omega)}{\omega} \cos \omega t \, d\omega. \tag{402}$$

Combining these two expressions,

$$A(t) = P(0) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Q(\omega)}{\omega} \cos \omega t \, d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega. \tag{403}$$

When

$$\left.\begin{aligned} P(-\omega) &= P(\omega) \\ Q(-\omega) &= -Q(\omega) \end{aligned}\right\} \tag{404}$$

which is true for most of the impedance functions commonly used, this may be written

$$A(t) = P(0) + \frac{2}{\pi}\int_{0}^{\infty} \frac{Q(\omega)}{\omega} \cos \omega t \, d\omega = \frac{2}{\pi}\int_{0}^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega. \tag{405}$$

These are valid for $t > 0$ and provided all the infinite integrals involved in the derivation converge. Now $A(t)$ is expressed explicitly by means of integrals of relatively simple form; and $A(t)$ can be found, at least except for the steady-state term, if either $P(\omega)$ or $Q(\omega)$ is known, provided all the infinite integrals formed from these and $\sin \omega t/\omega$ or $\cos \omega t/\omega$ converge, and provided the integrals can be evaluated after they have been written.

8. **Some General Considerations.** The results just obtained are not now very useful for computation, although they would immediately become very much so if a device were available for rapidly evaluating infinite integrals of this type. They do, however, tell much concerning the general interrelations of networks.

Expression (389) states in mathematical language that

the transient of a network is completely determined by the behavior of the generating function along the axis of imaginaries. In other words, the transient behavior of a network is determined as soon as its alternating-current steady-state performance can be expressed. Stated the other way, a knowledge of the response of the network to a simple unit constant voltage tells how the network will behave, not only when some other form of transient voltage is applied, but also for the alternating-current steady state.

The last equations go further. If conditions are such that the integrals converge, they state that complete knowledge of the circuit for all conditions is obtained if either the power-current characteristic in the alternating-current steady state is known, or the quadrature characteristic together with the zero-frequency current.

If two such networks are available which can be tested only at a certain number of external terminals; and if, when tested at these terminals, they are found to have the same power-current characteristic (expressed as a function of frequency), or the same quadrature characteristic and zero-frequency current, they are utterly indistinguishable by any electrical tests whatever carried on within the range for which the network parameters are constant.

9. **An Illustration.** For the simple resistance and inductance circuit, the generating function is

$$f(j\omega) = \frac{1}{R + jL\omega} = \frac{R}{R^2 + L^2\omega^2} - j\frac{L\omega}{R^2 + L^2\omega^2} \tag{406}$$

and hence

$$\left.\begin{aligned} P(\omega) &= \frac{R}{R^2 + L^2\omega^2} \\ Q(\omega) &= \frac{-L\omega}{R^2 + L^2\omega^2} \end{aligned}\right\}. \tag{407}$$

These power and quadrature characteristics are plotted in Fig. 38.

There are several points which may be noted in regard to

characteristics of this type as they are encountered in the usual networks. Notice that $Q(\omega)$ necessarily passes through the origin, that it cannot become infinite at any point when a network containing resistance is being considered, and that unless the network is impulsive it necessarily approaches

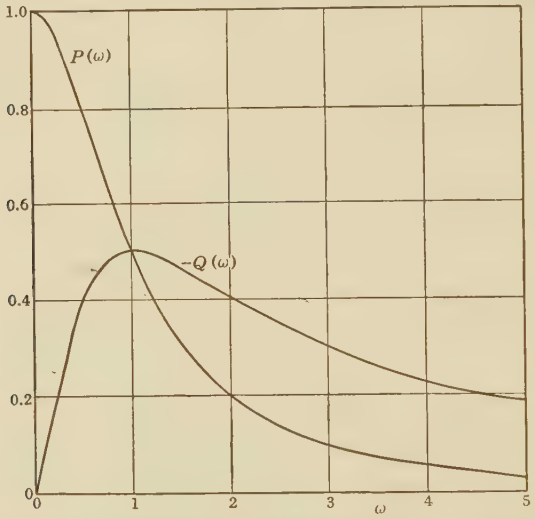

FIG. 38. Power and Quadrature Characteristics of Inductive Circuit for $R = L = 1$.

zero for high frequencies. $P(\omega)$ is zero or finite at $\omega = 0$, it cannot become infinite at any point, and it approaches zero or some finite value as the frequency is indefinitely increased. For this present example, the second form of (405) becomes

$$A(t) = \frac{2}{\pi}\int_0^\infty \frac{R}{\omega(R^2 + L^2\omega^2)} \sin \omega t \, d\omega. \tag{408}$$

This particular infinite integral has been evaluated[1]

$$\int_0^\infty \frac{\sin ax}{x(b^2 + x^2)}\, dx = \frac{\pi}{2\,b^2}(1 - \epsilon^{-ab}). \tag{409}$$

[1] Bierens de Haan, Nouvelles Tables D'Integrales Définies, p. 252, No. 1, p. 223, No. 5.

A comparison gives the familiar result:

$$A(t) = \frac{1}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right). \tag{410}$$

The same result can be obtained from the first form of (405), and

$$\int_0^\infty \frac{\cos ax}{b^2 + x^2}\, dx = \frac{\pi}{2b}\, \epsilon^{-ab}. \tag{411}$$

10. **Fourier Transforms.** One of the most convenient features of a Fourier integral expression is that it can be turned inside out.

From (384),

$$f(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon^{\omega t}\phi(\omega)\, d\omega \tag{412}$$

where

$$\phi(\omega) = \int_{-\infty}^{\infty} \epsilon^{-\omega\lambda} f(\lambda)\, d\lambda. \tag{413}$$

This is called the **Fourier transform.**[1] Written in terms of other variables it becomes:

$$\left.\begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos \lambda t F(\lambda)\, d\lambda \\ F(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos \lambda t f(\lambda)\, d\lambda \end{aligned}\right\}. \tag{414}$$

F is called the Fourier transform of f, and *vice versa*. Of course, before it can be applied, F must be of such nature that the integral converges, in which case f will be also

[1] E. C. Titchmarsh, A Contribution to the Theory of Fourier Transforms, Proc. London Math. Soc., **23**, 1923, p. 279, treats this transform in rigorous manner.

A remarkable list of Fourier transforms, useful in operational treatments is given in "The Practical Application of the Fourier Integral," by G. A. Campbell, *loc. cit.* Many of the transforms given there are not obtainable by the simple theory here reviewed, but may be obtained by the extended theory of the Fourier transform due to Wiener and Bochner.

convergent. These correspond to the form of the Fourier integral:

$$f(t) = \frac{2}{\pi}\int_0^\infty f(\lambda)\, d\lambda \int_0^\infty \cos \xi\lambda \cos \xi t\, d\xi. \tag{415}$$

There is an alternative form:

$$f(t) = \frac{2}{\pi}\int_0^\infty f(\lambda)\, d\lambda \int_0^\infty \sin \xi\lambda \sin \xi t\, d\xi. \tag{416}$$

The first of these will represent an even function, and the second, an odd function for any value of t. Either one will represent $f(t)$ correctly for positive t. To the latter corresponds the transform:

$$\left.\begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin \lambda t F(\lambda)\, d\lambda \\ F(t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sin \lambda t f(\lambda)\, d\lambda \end{aligned}\right\}. \tag{417}$$

We had

$$A(t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{f(\omega)}{\omega}\, \epsilon^{\omega t}\, d\omega. \tag{418}$$

The Fourier transform of this gives

$$\frac{f(\omega)}{\omega} = \int_{-\infty}^{\infty} \epsilon^{-\omega\lambda} A(\lambda)\, d\lambda \tag{419}$$

and if $A(t) = 0$ when $t < 0$,

$$\frac{f(\omega)}{\omega} = \int_0^\infty \epsilon^{-\omega\lambda} A(\lambda)\, d\lambda \tag{420}$$

which is the infinite integral theorem. In an expression of this sort $\frac{f(\omega)}{\omega}$ is called the **Laplacian transform** of A. Again, of course, the expression is valid only if the functions f and A are such that the integrals converge. If $A(-t) \neq 0$ we would need to retain the infinite lower limit. The use of this expression will result in the identification of p^{-1} with

$\int_{-\infty}^{t} dt$ rather than with $\int_{0}^{t} dt$. One case of this sort is met in Chapter XII. Of course when $A(t) = 0$ for $t < 0$, it makes no difference which we use. The treatment above was also limited to $f(0) = 0$. In case this is not so, the alternative expression:

$$A(t) - f(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{f(\omega) - f(0)}{\omega} \epsilon^{\omega t} \, d\omega \tag{421}$$

can be transformed to obtain

$$\frac{f(\omega)}{\omega} - \frac{f(0)}{\omega} = \int_{0}^{\infty} \epsilon^{-\omega\lambda} A(\lambda) \, d\lambda - \int_{0}^{\infty} \epsilon^{-\omega\lambda} f(0) \, d\lambda. \tag{422}$$

The second terms on each side cancel, giving

$$\frac{f(\omega)}{\omega} = \int_{0}^{\infty} \epsilon^{-\omega\lambda} A(\lambda) \, d\lambda \tag{420}$$

which now includes the case where $f(0)$ is not zero. By a similar transformation,

$$A(t) = \frac{2}{\pi} \int_{0}^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega \tag{423}$$

is changed into

$$\frac{P(\omega)}{\omega} = \int_{0}^{\infty} A(\lambda) \sin \omega\lambda \, d\lambda. \tag{424}$$

If $B(t)$ is written for the indicial admittance minus the steady-state term,

$$B(t) = A(t) - P(0) \tag{425}$$

then

$$B(t) = \frac{2}{\pi} \int_{0}^{\infty} \frac{Q(\omega)}{\omega} \cos \omega t \, d\omega \tag{426}$$

which transforms to

$$\frac{Q(\omega)}{\omega} = \int_{0}^{\infty} B(\lambda) \cos \omega\lambda \, d\lambda. \tag{427}$$

For the example of Sec. 9, equation (424) becomes

$$\frac{P(\omega)}{\omega} = \frac{1}{R}\int_0^\infty \left(1 - \epsilon^{\frac{-R\lambda}{L}}\right) \sin \omega\lambda \, d\lambda \tag{428}$$

and this integral can be evaluated by comparison with

$$\int_0^\infty \epsilon^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \tag{429}$$

and[1]

$$\lim_{a \to 0} \int_0^\infty \epsilon^{-ax} \sin bx \, dx = \frac{1}{b} \tag{430}$$

to yield a correct expression for $\frac{P(\omega)}{\omega}$.

Similarly, from (427),

$$\frac{Q(\omega)}{\omega} = -\frac{1}{R}\int_0^\infty \epsilon^{-\frac{R\lambda}{L}} \cos \omega\lambda \, d\lambda \tag{431}$$

which can be evaluated from

$$\int_0^\infty \epsilon^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}. \tag{432}$$

Expressions (424) and (427) enable us to evaluate directly the power and quadrature characteristics of a network when its indicial admittance is known. They also serve as simple expressions for checking the validity of an operational process in much the same way as the infinite integral theorem. Like all expressions involving infinite integrals they are none too readily evaluated in cases that are at all complicated.

11. **Summary.** The above is the simple portion of the solution of networks by means of Fourier integrals.

Direct operational methods may be regarded as shorthand processes of evaluating and tabulating the results of Fourier analysis. They are much more convenient and compact

[1] This is a "summable" integral, and is evaluated by an extension of the usual treatment. Hardy, Quarterly Journal, **35**, 1903, p. 22. Bromwich, Theory of Infinite Series, Macmillan, London, 1908, p. 496.

than the latter, and where no new features are involved, enable us to work rapidly and by the use of only very simple mathematics. The establishment of new operational methods, the extension to problems with novel features, requires every time that we fall back upon the rigorous treatment for substantiation.

PROBLEMS ON CHAPTER X

Prob. 1–10. Sketch a curve showing the conductance or power-current characteristic of an R and C series circuit as a function of frequency. Do the same for its susceptance or quadrature-current characteristic.

Prob. 2–10. By examining the conductance and susceptance characteristics of other elementary circuits consisting of R, L, and C in various series and parallel arrangements, what generalizations can you make regarding the intercepts, continuity, general form, and ultimate values for large frequencies, of each of these two current characteristics?

Prob. 3–10. Find the indicial admittance of the simple R and C series circuit by means of each of the two equations which express the indicial admittance of a network in terms of its conductance and susceptance characteristics.

Prob. 4–10. Construct two networks having the same quadrature characteristics as given by $Q(\omega)$, but different power characteristics given by $P(\omega)$. Consider equation (405) and state how the values of the indicial admittance of the two networks will differ.

Prob. 5–10. Can two networks have the same indicial admittance, and the same power-current characteristics, but different quadrature-current characteristics? Examine by means of equations (424) and (427).

Prob. 6–10. Evaluate and plot: $\tanh \frac{kp}{2} (1 + \epsilon^{-kp})\, 1$.

Prob. 7–10. Plot $\frac{1}{2}\left(1 + \tanh \frac{sp}{2}\right) 1$. Develop this same expression by writing a series of terms similar in form to that of Prob. 6–10, and adding.

Prob. 8–10. Develop the operational expression for the triangular wave shape constructed by a succession of 45-degree lines making a succession of right equilateral triangles. Compare with the expression obtained by applying p^{-1} to a rectangular wave.

Form the operational expression for one such triangle (a) located with one corner at the origin, (b) located with the nearest corner a distance h from the origin.

Prob. 9–10. Evaluate $\left(\frac{p}{p+b}\tanh\frac{kp}{2}\right)1$. To what physical problem does it correspond?

Prob. 10–10. From the known indicial admittance of a circuit containing resistance and capacitance in series, what infinite integrals can be evaluated?

Prob. 11–10. Assuming that the infinite integrals which occur could be readily evaluated, for example by some graphical process, and that the power-current characteristic of a network is known, how could its quadrature-current characteristic be determined?

Prob. 12–10. Write the Fourier integral expression for the response of an R, L, and C series circuit to a sinusoidal e.m.f. Simplify as far as possible.

Prob. 13–10. Evaluate by operational methods several of the Fourier transforms given by Campbell, "The Practical Application of the Fourier Integral." B.S.T.J., October, 1928.

Prob. 14–10. Find the operational expression $\frac{1}{Z(p)}1$ which gives a periodic time function consisting of a succession of positive half sinusoids extending to the right of $t = 0$.

Prob. 15–10. Find the operational expression for a time function constructed from that of Prob. 14–10 by reducing every odd half sinusoid to zero.

Prob. 16–10. Suggest several additional operational formulations similar to those contemplated in Probs. 6, 7, 8, 9, 14, 15.

CHAPTER XI

FRACTIONAL-ORDER DERIVATIVES

Consider a cable of capacitance c and resistance r each per unit length, the other parameters being negligible. The differential equations are

$$\left.\begin{aligned} \frac{\partial i}{\partial x} &= -cpe \\ \frac{\partial e}{\partial x} &= -ri \end{aligned}\right\} \tag{433}$$

or

$$\left.\begin{aligned} \frac{\partial^2 e}{\partial x^2} &= rcpe \\ \frac{\partial^2 i}{\partial x^2} &= rcpi \end{aligned}\right\}. \tag{434}$$

Put

$$\beta^2 = rcp \tag{435}$$

and formal solutions of these, treating p as a constant, are[1]

$$\left.\begin{aligned} e &= \epsilon^{-\beta x}K_1 + \epsilon^{\beta x}K_3 \\ i &= \epsilon^{-\beta x}K_2 + \epsilon^{\beta x}K_4 \end{aligned}\right\}. \tag{436}$$

Consider now that voltage is applied only at $x = 0$ and that the cable extends a long distance in the positive x-direction. This is called an infinite cable. The current and voltage then fall off as x becomes very large, so that it appears reasonable to set $K_3 = K_4 = 0$, and

$$\left.\begin{aligned} e &= \epsilon^{-\beta x}K_1 \\ i &= \epsilon^{-\beta x}K_2 \end{aligned}\right\}. \tag{437}$$

[1] Note that we write $e = \epsilon^{-\beta x}K_1$ rather than $e = K_1\epsilon^{-\beta x}$ because $\epsilon^{-\beta x}$ is here an operator acting on the time function K_1.

If e_0 and i_0 are the voltage and current at $x = 0$,

$$\left.\begin{aligned} e &= \epsilon^{-\beta x} e_0 \\ i &= \epsilon^{-\beta x} i_0 \end{aligned}\right\}. \tag{438}$$

Now, from $\dfrac{\partial e}{\partial x} = -ri$,

$$-\beta \epsilon^{-\beta x} e_0 = -r\epsilon^{-\beta x} i_0 \tag{439}$$

or

$$i_0 = \frac{\beta}{r} e_0 = \sqrt{\frac{cp}{r}}\, e_0 . \tag{440}$$

The driving-point impedance of an infinite cable is thus

$$Z(p) = \sqrt{\frac{r}{cp}} = \sqrt{\frac{r}{c}}\, p^{-\frac{1}{2}} \tag{441}$$

and the transfer impedance for a point x units down the cable is

$$Z_{0x}(p) = \epsilon^{\beta x} \sqrt{\frac{r}{cp}} = \epsilon^{\sqrt{rcx^2 p}} \sqrt{\frac{r}{c}}\, p^{-\frac{1}{2}} . \tag{442}$$

The same equations apply, of course, to parallel heat flow in a thick block of material. There is thus encountered in a simple physical problem the idea of a **fractional-order derivative.**

2. **Various Simple Operators Involving Fractional Powers of p. The Gamma Function.** We are called upon to interpret the operational expression:

$$p^{\frac{1}{2}}\ 1.$$

The corresponding infinite integral is

$$a^{-\frac{1}{2}} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda \tag{443}$$

which is satisfied by

$$A(\lambda) = \frac{1}{\sqrt{\pi\lambda}} \tag{444}$$

as can be seen as follows: Replace λ by x^2, obtaining

$$\int_0^\infty \epsilon^{-a\lambda} \frac{1}{\sqrt{\pi\lambda}}\, d\lambda = \frac{2}{\sqrt{\pi}} \int_0^\infty \epsilon^{-ax^2}\, dx. \tag{445}$$

This is the probability integral, the value of which is $\frac{1}{2}\sqrt{\frac{\pi}{a}}$, so that the entire expression reduces to $1/\sqrt{a}$ and establishes the formula[1]:

$$p^{\frac{1}{2}}\, 1 = \frac{1}{\sqrt{\pi t}}. \tag{446}$$

Heaviside established this expression by comparison with the known solution of the heat-flow problem obtained by classical methods.[2]

When unit e.m.f. is applied to an infinite cable, the current is then

$$i_0 = \sqrt{\frac{c}{\pi r}}\, t^{-\frac{1}{2}}. \tag{447}$$

The expression $p^{\frac{1}{2}}\, 1$ thus represents the current at the home end of an infinite cable due to the application of unit e.m.f., the cable parameters having the numerical relation $c = r$.

The ambiguities of sign which occur in this, and other sections, will be discussed in Chapter XIII.

The charge that enters the cable is

$$q_0 = p^{-1}\, i_0 = \sqrt{\frac{c}{\pi r}}\, p^{-1} t^{-\frac{1}{2}}\, 1 = \sqrt{\frac{c}{r}}\, p^{-\frac{1}{2}}\, 1. \tag{448}$$

To the operational expression:

$$p^{-\frac{1}{2}}\, 1$$

[1] For an extension of a definition of the derivative by means of the Cauchy contour integral, and its use on $p^{\frac{1}{2}}1$, see V. Bush, Trans. A.I.E.E., **38**, 1919, p. 468.

See N. Wiener, The Operational Calculus, Mathematische Annalen, Band 95, 1926, p. 557, Publications of M.I.T., Series II, No. 109.

[2] For a solution of this cable problem by classical methods see, for example, J. H. Jeans, Electricity and Magnetism, Cambridge University Press, 1923, p. 332.

corresponds the infinite integral equation

$$\frac{1}{a^{\frac{1}{2}}} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda \tag{449}$$

and this is satisfied by

$$A(\lambda) = \frac{2}{\sqrt{\pi}} \lambda^{\frac{1}{2}}. \tag{450}$$

For,

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \epsilon^{-a\lambda} \lambda^{\frac{1}{2}}\, d\lambda$$

integrated by parts, gives

$$\frac{1}{a\sqrt{\pi}} \int_0^\infty \epsilon^{-a\lambda} \lambda^{-\frac{1}{2}}\, d\lambda = \frac{1}{a^{\frac{1}{2}}} \tag{451}$$

so

$$p^{-\frac{1}{2}}\, 1 = \frac{1}{\sqrt{\pi}}\, p^{-1} t^{-\frac{1}{2}}\, 1 = \frac{2}{\sqrt{\pi}}\, t^{\frac{1}{2}}\, 1. \tag{452}$$

The charge that enters the cable during any finite time is thus finite although the current is infinite at the first instant.

Notice that the operational formula above agrees with the identification of p^{-1} with $\int_0^t dt$, although the operand $t^{-\frac{1}{2}}\, 1$ is infinite at $t = 0$. Likewise the operator p^{-h} operating on the same operand can be interpreted as

$$p^{-h} t^{-\frac{1}{2}}\, 1 = \frac{t^{h-\frac{1}{2}}}{\frac{1}{2} \cdot \frac{3}{2} \cdots (h - \frac{1}{2})} \tag{453}$$

where h is a positive integer, or more generally as

$$p^{-h} t^n\, 1 = \frac{t^{h+n}}{(1+n)\,(2+n) \cdots (h+n)} \tag{454}$$

where the exponent of t has been considered to be any half-integral value.

It will be recalled that (209) was written

$$p^m t^n \, 1 = \frac{n! t^{n-m}}{(n-m)!} \tag{209}$$

where m and n were integers, and $n - m > 0$. If in this expression the factorials are interpreted in accordance with the relation

$$n! = \Gamma(n+1) \tag{455}$$

in which Γ signifies the **gamma function**, it will be seen by comparison with (454) that (209) can now be extended to cases in which the exponent of t has half-integral values. Thus (209) can be written as

$$p^m t^n \, 1 = \frac{\Gamma(n+1) t^{n-m}}{\Gamma(n-m+1)} \tag{456}$$

in which the limitations now are that $\Gamma(n+1)$ and $\Gamma(n-m+1)$ exist.

The gamma function is defined by the integral

$$\Gamma(n) = \int_0^\infty x^{n-1} \epsilon^{-x} \, dx. \tag{457}$$

In this expression n is limited to values greater than zero, but it will be shown in (460) below how the gamma functions of negative values can be obtained. A plot of these functions is shown in Fig. 39.

The relation between the gamma function and the factorial, (455), holds immediately when the former is defined by the integral above. For an integral value, the factorial is defined by the usual rule:

$$m! = 1 \cdot 2 \cdot 3 \ldots m. \tag{458}$$

For a fraction, this definition of the factorial is utterly without meaning; and a more general definition is required if the concept of the **factorial of a fraction** is to be utilized. It was of course possible, when this question first arose, to define factorials of fractions in any way whatever. The

choice of a definition of a new mathematical concept is a question of utility and consistency. Therefore the scheme adopted was to define the factorial of a fraction in terms of the gamma function, that is, to assume that relation (455) holds also when n is non-integral. The convenience in our

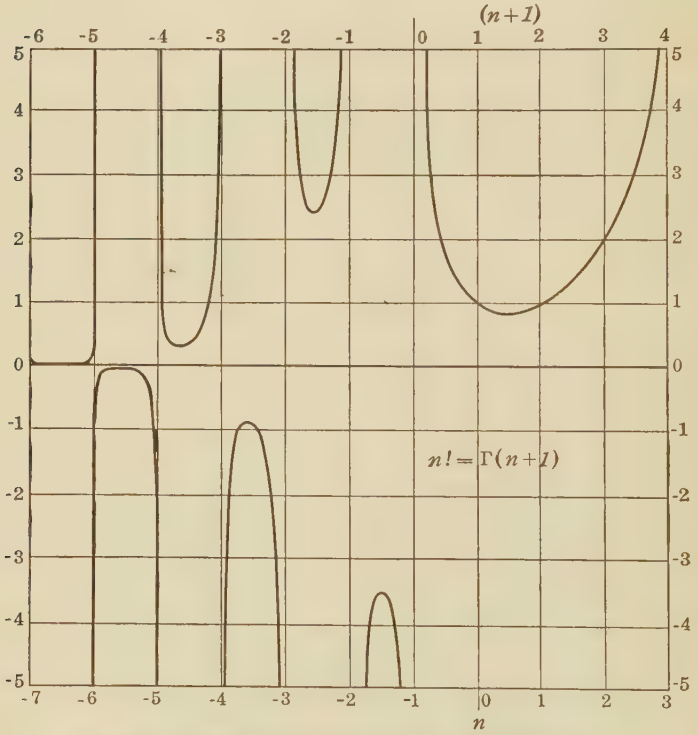

FIG. 39. The Gamma Function and the Factorial.

case is apparent, for it enables us to extend a formula for integers to the case of non-integers without any need for change of form or supplementary notes.

The same matter of expediency of definition has often arisen. Thus when the idea of a derivative to a fractional order was first met in mathematics there was no reason that

it should have been defined in one way rather than another, except the very potent reason of the convenience or lack of convenience of the resulting mathematical system. It was best defined therefore in such manner that the rules for ordinary derivatives could be carried over to the new concept with as little alteration as possible. However, in analyzing a problem, the results obtained by the use of these rules under the extended definition had to be examined before they could be considered as valid.[1]

This is a good place to emphasize again that the definition of operators applied to discontinuous functions is an entirely similar matter. They should be useful in solving certain types of problems, and they are hence defined by comparison with some classical scheme of solving problems of the same type. There are several ways that this can be done and has been done. Of course they will all lead to the same result when the operators arise from problems of that certain type, if no errors are made in manipulation; but there is no guarantee whatever that they will lead to even similar results if applied to operators which do not arise from problems of the type contemplated in setting up the definitions. This will be seen more definitely in Chapter XIII. The operators have been defined in terms of an integral in the complex plane by Bromwich, Wagner, Wiener and others, but the exact path for the integral was not always the same. They are defined by Carson by a real infinite integral. Volterra defines them in terms of his integral equations. Applied to the usual network problem, these schemes all lead to the same correct result. Applied to an unusual problem, such as that of a network with a negative resistance, or applied to a generating function set up at random without reference to a network, they lead to widely different results, as would be expected. Of course that definition is best for

[1] S. Pincherle, Funktionaloperationen und-gleichungen, Enc. Math. Wiss., II; or Équations et Opérations Functionnelles, Enc. des Sciences Mathematiques, II, 5, 1.

ordinary purposes which is most simple, and for extraordinary purposes which is most inclusive.

To return to the matter in hand, however, the following values of $\Gamma(n)$

$$\Gamma(1) = 1$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
$$\Gamma(0) = \pm\infty$$

give the values of the extended factorial:

$$0! = 1$$
$$-\tfrac{1}{2}\,! = \sqrt{\pi}$$
$$-1\,! = \pm\infty\,.$$

The following relations are also useful:

$$\Gamma(n+1) = n\Gamma(n) \tag{459}$$

so

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}$$
$$\Gamma(\tfrac{5}{2}) = \tfrac{1}{2}\cdot\tfrac{3}{2}\sqrt{\pi}, \text{ etc.}$$

A slight rearrangement of (459) makes possible the extension of the definition of the gamma function to cases of negative numbers, for

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}\,. \tag{460}$$

Thus

$$\Gamma(-\tfrac{1}{2}) = -2\,\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}$$
$$\Gamma(-\tfrac{3}{2}) = -\tfrac{2}{3}\,\Gamma(-\tfrac{1}{2}) = \tfrac{2}{3}\cdot\tfrac{2}{1}\sqrt{\pi}$$
$$\Gamma(-\tfrac{5}{2}) = -\tfrac{2}{5}\,\Gamma(-\tfrac{3}{2}) = -\tfrac{2}{5}\cdot\tfrac{2}{3}\cdot\tfrac{2}{1}\sqrt{\pi}, \text{ etc.}$$

Now note that if in (456) n is made zero and m is made $\frac{1}{2}$, which gives the operator $p^{\frac{1}{2}}\,1$, the relation (446) is obtained; and if m is made $-\frac{1}{2}$, which gives the operator $p^{-\frac{1}{2}}\,1$, the relation (452) is obtained.

Equation (456) can also be used to interpret $p^{\frac{3}{2}}\,1$ or in general $p^k\,1$ where k is a half integer greater than zero. Thus (456) reduces to

$$p^{\frac{3}{2}}\,1 = \frac{\Gamma(1)t^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2})} \tag{461}$$

which gives, since $\Gamma(1) = 1$ and $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$,

$$-\frac{1}{2\sqrt{\pi}}t^{-\frac{3}{2}}.$$

Note that interpreting this operator as $p \cdot p^{+\frac{1}{2}}\,1$ gives the same result, since from (446)

$$p^{\frac{1}{2}}\,1 = \frac{1}{\sqrt{\pi t}}$$

and hence

$$p^{\frac{3}{2}}\,1 = p \cdot p^{\frac{1}{2}}\,1 = -\frac{1}{2\sqrt{\pi}}t^{-\frac{3}{2}}. \tag{462}$$

Thus, as in the discussion of (452), this result agrees with the identification of p with $\frac{d}{dt}$ in this case, although the operand $t^{-\frac{1}{2}}\,1$ is infinite at $t = 0$.

In general, (456) will evaluate $p^m\,1$ by making $n = 0$, giving

$$\begin{aligned} p^m\,1 &= \frac{\Gamma(1)t^{-m}}{\Gamma(-m+1)} \\ &= \frac{t^{-m}}{\Gamma(1-m)} \end{aligned} \tag{463}$$

except for positive integral values of m. This general operational formula can be checked by substitution in the infinite integral equation as follows:

$$a^{m-1} = \frac{1}{\Gamma(1-m)}\int_0^\infty \epsilon^{-a\lambda}\lambda^{-m}\,d\lambda. \tag{464}$$

The evaluation of this integral from a table of infinite integrals will give

$$\frac{\Gamma(-m+1)}{a^{-m+1}}$$

which multiplied by $1/\Gamma(1-m)$ reduces the equation to an identity.

3. **Graphical Interpretation.** The concept of fractional-order integration and differentiation is an interesting one. Fig. 40 is a plot of the various curves obtained from (463) if m, for convenience, is given a range of values from -2 to $+\frac{1}{2}$. Since the gamma function in the expression exists for all real values of m except the positive integers, it appears

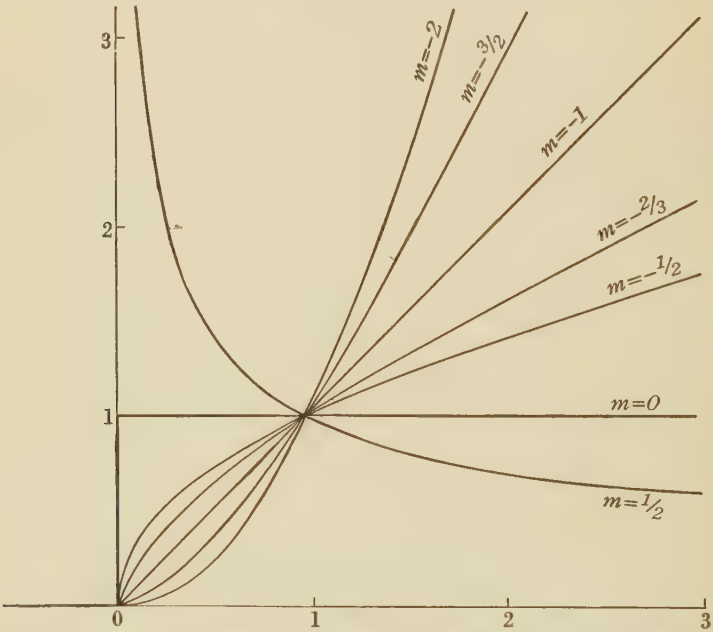

FIG. 40. Nest of Curves $\Gamma(1-m)p^m 1$. Fractional-order Derivatives and Integrals of Unit Function.

that a complete family of curves, gradually deformed, will be obtained as m takes on all these real values.

4. **The Infinite Line. Bessel Functions.** It seems to be true that problems on electric transmission lines always involve p under a radical. Consider, for example, the infinite line in which the resistance, inductance, and capacitance per unit length are r, l, and c respectively, the leakage parameter g

being negligible. The generating function is now

$$\frac{1}{Z(p)} = \sqrt{\frac{cp}{r + lp}}. \tag{465}$$

This may be derived from the differential equations as before, or it may be obtained by replacing $j\omega$ by p in the familiar expression for the alternating-current impedance of a line. Although this process of obtaining a generating function has been justified for finite, lumped-parameter networks, its extension to cases of distributed parameters needs to be examined. This can be done by retracing the steps used to establish the Fourier integral, now considering the arbitrary function $f(t)$ as the e.m.f. which is to be applied to the line. Thus this e.m.f. is expressed as an infinite sum of steady-state sinusoidal voltages each of which, when divided by the alternating-current impedance $Z(j\omega)$ of the line, produces its respective steady-state current. In finally establishing the Fourier integral for the response, the alternating-current impedance function becomes the reciprocal of the generating function $f(\omega)$ as written in (386). Thus $\frac{1}{Z(p)}$ can be formed at once by replacing $j\omega$ by p in the alternating-current impedance of the line.

As this process provides an independent means of establishing the operator for the line, comparison of the result obtained directly from the differential equations with the result obtained in this way gives further justification for dropping the positive exponential terms as was done in Sec. 1.

Consider now the problem of the entering current in a long aerial line or loaded cable. Use the abbreviation

$$\alpha = \frac{r}{2l} \tag{466}$$

Then

$$i_0(t) = \frac{1}{Z(p)} e_0 = \sqrt{\frac{c}{l}} \frac{1}{\sqrt{1 + \frac{2\alpha}{p}}} e_0 \tag{467}$$

so for unit applied e.m.f. we are called on to interpret

$$\frac{1}{\sqrt{1+\frac{2\alpha}{p}}}\,1. \tag{468}$$

We shall first treat

$$\frac{p}{\sqrt{p^2+\alpha^2}}\,1. \tag{469}$$

This corresponds to the integral:

$$\frac{1}{\sqrt{a^2+\alpha^2}} = \int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda \tag{470}$$

and a solution of this can be obtained from tables of infinite integrals, as

$$A(\lambda) = J_0(\alpha\lambda). \tag{471}$$

In this expression J_0 is the Bessel function of order zero.[1] This is defined by the series,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \cdots \tag{472}$$

and the fact that this is a solution can be seen by performing the integration:

$$\int_0^\infty \epsilon^{-a\lambda}\left(1 - \frac{\alpha^2\lambda^2}{2^2} + \frac{\alpha^4\lambda^4}{2^2 4^2} - \cdots\right) d\lambda$$

term by term, and noting that the result is the binomial expansion of $(a^2+\alpha^2)^{-\frac{1}{2}}$. So

$$\frac{p}{\sqrt{p^2+\alpha^2}}\,1 = J_0(\alpha t). \tag{473}$$

[1] G. N. Watson, Theory of Bessel Functions, Cambridge University Press, 1922. Gray, Mathews, and Macrobert, Bessel Functions, Macmillan, London, 1922. Tables and graphs of these functions may also be found in Jahnke und Emde, Funktionentafeln mit Formeln und Kurven, Teubner, Berlin, 1923.

Replace α by $j\alpha$, and

$$\frac{p}{\sqrt{p^2 - \alpha^2}}\, 1 = J_0(j\alpha t). \tag{474}$$

There is a peculiarity here, for the generating function has a pole to the right of the axis of imaginaries. This will be disregarded for the present, and will be brought up again in Chapter XIII.

The expression on the right of (474) is real, in spite of the presence of j. It corresponds to the series:

$$J_0(jx) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \cdots. \tag{475}$$

This series can be used to check the formula. In the same way a general expression:

$$\frac{p}{\sqrt{p^2 + \alpha^2}}\left(\frac{\sqrt{p^2 + \alpha^2} - p}{\alpha}\right)^n 1 = J_n(\alpha t) \tag{476}$$

can be obtained.

Here J_n is the nth-order Bessel function defined by

$$J_n(x) = \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right]. \tag{477}$$

It should be remarked that there is nothing mysterious about functions defined by series in this manner. The definition of the sine of x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

is the same sort of thing. Bessel functions and similar developments are less familiar, for they are less used and the laws governing their combination are less simple. But they are no more bizarre, only somewhat more complicated than the usual functions.

Now apply the shifting process (280) to the operator of our problem.

$$\frac{1}{\sqrt{1+\frac{2\alpha}{p}}}1 = \epsilon^{-\alpha t}\frac{p}{p-\alpha}\frac{1}{\sqrt{1+\frac{2\alpha}{p-\alpha}}}1$$

$$= \epsilon^{-\alpha t}\frac{p}{\sqrt{p^2-\alpha^2}}1$$

$$= \epsilon^{-\alpha t}J_0(j\alpha t). \tag{478}$$

The solution of the line problem which introduced these operators is then

$$i_0 = E_0\sqrt{\frac{c}{l}}\epsilon^{-\alpha t}J_0(j\alpha t) \tag{479}$$

where E_0 is the magnitude of the suddenly-applied e.m.f. This solution is an important one in the theory of transmission line transients. It not only gives the current entering an infinite line, but it gives the current entering a finite line with any sort of termination at the distant end up to the time of the arrival of the first reflected wave. This follows because the conditions at the distant end cannot affect matters at the home end until a time has elapsed sufficient for a wave to run down the line and back.

This solution has been checked experimentally on artificial transmission lines and also on an actual power line.[1] If E_0 is taken as unity this solution becomes the indicial admittance of the line. As such, it has been used to obtain the solution for a suddenly-applied alternating e.m.f. by mechanical integration, and the result checked experimentally.

The agreement between computation and measurement is shown in Fig. 41 and Fig. 42 for the home-end current on 662 miles of single-phase artificial line, for closure of the switch at two different points of the voltage wave.

[1] V. Bush, Transmission Line Transients, M.I.T., E.E. Research Bull. No. 50. A discussion of the practical bearing of many of the solutions obtained is outside of the scope of this present text.

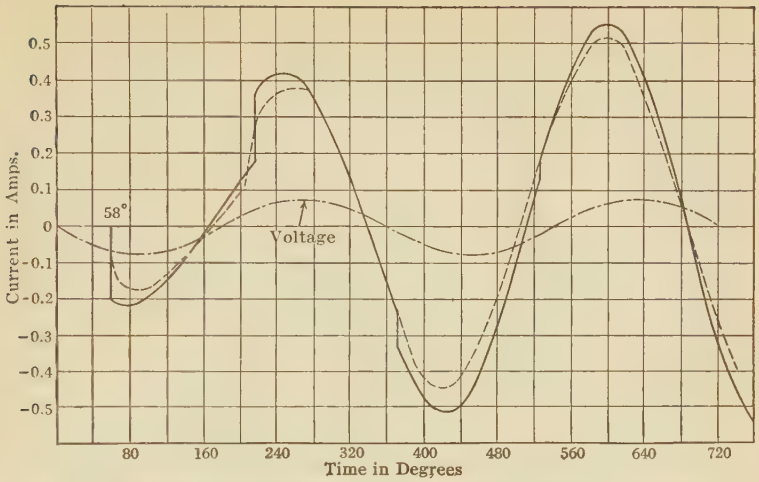

FIG. 41. Calculated and Measured Alternating-Current Transients of an Artificial Transmission Line. (Transmission Line Transient Investigation. M.I.T., E.E. Research Bulletin No. 50.)

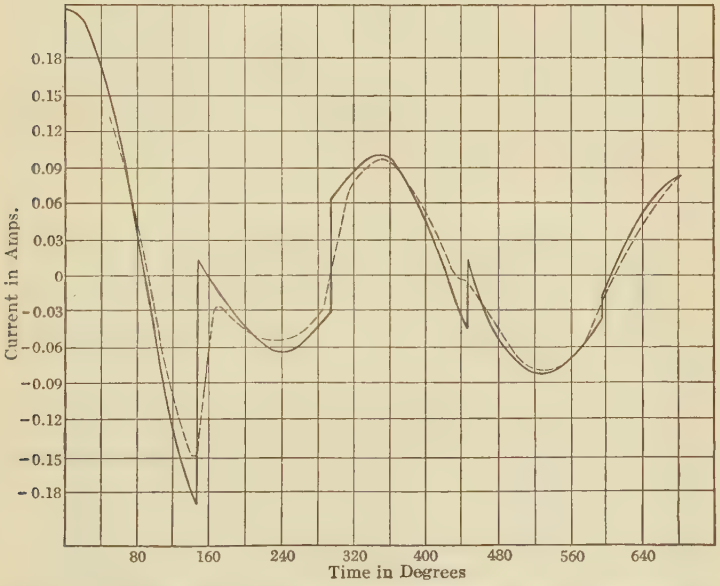

FIG. 42. Calculated and Measured Alternating-Current Transients of an Artificial Transmission Line. (Transmission Line Transient Investigation. M.I.T., E.E. Research Bulletin No. 50.)

5. **Other Operators. A Heat-flow Problem.** Other simple combinations involving p under the radical may be set up and evaluated. Thus

$$\frac{p}{\sqrt{p+b}}\,1 \tag{480}$$

may be transformed by the shifting process into

$$\epsilon^{-bt}\frac{p}{p-b}\frac{p-b}{\sqrt{p}}\,1 = \epsilon^{-bt}p^{\frac{1}{2}}\,1. \tag{481}$$

So

$$\frac{p}{\sqrt{p+b}}\,1 = \epsilon^{-bt}\frac{1}{\sqrt{\pi t}}. \tag{482}$$

Apply p^{-1} to this:

$$\frac{1}{\sqrt{p+b}}\,1 = \int_0^t \frac{\epsilon^{-bt}}{\sqrt{\pi t}}\,dt. \tag{483}$$

This integral cannot be evaluated except in a series. Successive integrations by parts will give

$$\frac{1}{\sqrt{p+b}}\,1 = 2\sqrt{\frac{t}{\pi}}\,\epsilon^{-bt}\left[1 + \frac{2\,bt}{1\cdot 3} + \frac{(2\,bt)^2}{1\cdot 3\cdot 5} + \cdots\right] \tag{484}$$

which is a convergent series. The integral form is, however, usually more convenient. It is more compact, a larger amount of information can be obtained by studying it, and for computation purposes it can be evaluated mechanically if necessary.

There is an interesting point here. We state that we have encountered an integral which cannot be evaluated. What we mean is that it cannot be evaluated in terms of functions with which we are familiar and of which we have convenient tables. It is no different inherently from other integrals. It simply happens that the series involved in its evaluation has not received a special name and been computed for reference.

Also consider

$$\sqrt{p+b}\ 1.$$

Write this

$$\frac{p+b}{\sqrt{p+b}}\ 1 = \left(\frac{p}{\sqrt{p+b}} + \frac{b}{\sqrt{p+b}}\right) 1 \qquad (485)$$

and its parts are recognizable. So

$$\sqrt{p+b}\ 1 = \epsilon^{-bt}\frac{1}{\sqrt{\pi t}} + b\int_0^t \frac{\epsilon^{-bt}}{\sqrt{\pi t}}\,dt. \qquad (486)$$

Notice the striking fact that there is little connection between the simplicity of an operational formula and its corresponding time function. These expressions occur in the problem of the leaky cable, with parameters r, c, and g, the latter being the leakage conductance per unit of length. The inductance is considered negligible.

This problem is of more importance in heat flow than it is in electricity. It there corresponds to heat flow in a rod, where heat is being lost from the sides.[1]

The operational equation is

$$i_0(t) = \sqrt{\frac{cp+g}{r}}\,e_0 \qquad (487)$$

and for a suddenly-applied e.m.f of constant value E_0, this becomes, on putting $b = \frac{g}{c}$,

$$\begin{aligned} i_0(t) &= E_0\sqrt{\frac{c}{r}}\sqrt{p+b}\ 1 \\ &= E_0\sqrt{\frac{c}{r}}\left\{\epsilon^{-bt}\frac{1}{\sqrt{\pi t}} + b\int_0^t \frac{\epsilon^{-bt}}{\sqrt{\pi t}}\,dt\right\}. \end{aligned} \qquad (488)$$

[1] For the discussion of a large number of heat-flow problems, see H. S. Carslaw, Math. Theory of Conduction of Heat in Solids, Macmillan, 1921; T. J. I'A. Bromwich, Symbolical Methods in the Theory of Conduction of Heat, Proc. Cambridge Phil. Soc., **20**, 1921, p. 411.

If $g = 0$ this reduces to the simple expression for the non-leaky cable. Note the effect of leakage is to apply an exponential and to add a term which is proportional to the leakage and the integral of the first term.

6. **The Four-parameter Line.** When all four parameters are present, the operational formula is

$$i_0(t) = \sqrt{\frac{cp + g}{lp + r}}\, e_0. \tag{489}$$

Use the abbreviations:

$$\left.\begin{aligned} \rho &= \frac{r}{2l} + \frac{g}{2c} \\ \sigma &= \frac{r}{2l} - \frac{g}{2c} \\ v &= \frac{1}{\sqrt{lc}} \end{aligned}\right\} \tag{490}$$

and for constant applied e.m.f.,

$$i_0(t) = E_0 \frac{cp + g}{lcvp} \frac{p}{\sqrt{(p + \rho)^2 - \sigma^2}}\, 1. \tag{491}$$

To evaluate

$$\frac{p}{\sqrt{(p + \rho)^2 - \sigma^2}}\, 1$$

shift and obtain

$$\epsilon^{-\rho t} \frac{p}{\sqrt{p^2 - \sigma^2}}\, 1$$

and thus

$$\frac{p}{\sqrt{(p + \rho)^2 - \sigma^2}}\, 1 = \epsilon^{-\rho t} J_0(j\sigma t). \tag{492}$$

The second term is obtained by integrating the first, and so

$$i_0(t) = E_0 \sqrt{\frac{c}{l}}\, \epsilon^{-\rho t} J_0(j\sigma t) + E_0 g v \int_0^t \epsilon^{-\rho t} J_0(j\sigma t)\, dt. \tag{493}$$

This is the complete expression for entering current on a transmission line. If $\sigma = 0$, that is, if

$$\frac{r}{l} = \frac{g}{c}$$

then $J_0 = 1$, and (493) reduces, on performing the integration, to

$$i_0(t) = E_0\sqrt{\frac{c}{l}}. \tag{494}$$

This is the case of the distortionless line, so called. The entering current is steady and is obtained by multiplying the applied voltage by $\sqrt{\frac{c}{l}}$, which is sometimes called the surge admittance. We shall have this same steady value of entering current if there is no loss; that is, if $r = g = 0$.

7. **Cable with Terminal Condenser.** Another simple form of operator involving $p^{\frac{1}{2}}$ may be put in any of the forms:

$$\frac{\sqrt{p}}{\sqrt{p}+\sqrt{b}}\,1 = \frac{1}{1+\sqrt{\frac{b}{p}}}\,1 = \frac{p}{p-b}\left(1-\sqrt{\frac{b}{p}}\right)1$$

$$= \left(\frac{p}{p-b} - \frac{1}{p}\,\frac{p}{p-b}\,\sqrt{bp}\right)1. \tag{495}$$

The last expression is in form for the use of Borel's theorem, for each component part can be recognized.
So

$$\frac{1}{p}\,\frac{p}{p-b}\,\sqrt{bp}\,1 = \sqrt{\frac{b}{\pi}}\int_0^t \epsilon^{b(t-\lambda)}\lambda^{-\frac{1}{2}}\,d\lambda$$

$$= \sqrt{\frac{b}{\pi}}\,\epsilon^{bt}\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{\lambda}}\,d\lambda \tag{496}$$

and

$$\frac{1}{1+\sqrt{\frac{b}{p}}}\,1 = \epsilon^{bt}\left(1-\sqrt{\frac{b}{\pi}}\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{\lambda}}\,d\lambda\right). \tag{497}$$

This integral can be expressed in a series by successive integration by parts, giving as in (484)

$$\frac{1}{1+\sqrt{\frac{b}{p}}}1 = \epsilon^{bt} - 2\sqrt{\frac{bt}{\pi}}\left(1+\frac{2\,bt}{1\cdot 3}+\frac{(2\,bt)^2}{1\cdot 3\cdot 5}+\cdots\right) \quad (498)$$

which is convergent and is easy to compute from, when t is small. In Chapter XIII it will be seen that here in our evaluation we have split a retrospective operator into parts which are not retrospective. There is no harm in this provided our treatment is consistent, although the use of Borel's theorem in this connection needs more comment than we have given here. However, this last result will be checked presently by a series expansion of the operator, which provides justification in this particular problem.

The problem of an infinite cable with a terminal condenser C_0 illustrates this operator. The current entering the cable is

$$i_0 = \sqrt{\frac{cp}{r}}\,e_0 \quad (499)$$

where e_0 is the voltage at the cable terminals. Another expression for the current is given by the impedance of the condenser.

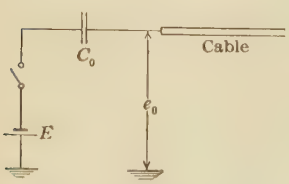

FIG. 43. Infinite Cable with Terminal Condenser.

$$i_0 = C_0 p(E - e_0) \quad (500)$$

where E is the applied e.m.f., Fig. 43. Thus

$$e_0 = \frac{pC_0}{pC_0 + \sqrt{\frac{cp}{r}}}E1 \quad (501)$$

and if the abbreviation

$$\sqrt{b} = \frac{1}{C_0}\sqrt{\frac{c}{r}} \quad (502)$$

is used, this becomes

$$e_0 = E\frac{1}{1+\sqrt{\frac{b}{p}}}1. \tag{503}$$

So

$$e_0 = E\epsilon^{\frac{ct}{C_0^2 r}}\left(1 - \frac{1}{C_0}\sqrt{\frac{c}{\pi r}}\int_0^t \frac{\epsilon^{-\frac{c\lambda}{C_0^2 r}}}{\sqrt{\lambda}}\,d\lambda\right). \tag{504}$$

The entering current can now be obtained, if it is wanted, from

$$i_0 = C_0\frac{d}{dt}(E - e_0) \tag{505}$$

for $(E - e_0)$ is zero at $t = 0$.

8. **The Half-power Exponential Operator.** In equation (438) was given an expression:

$$e = \epsilon^{-\beta x}e_0 \tag{506}$$

for the voltage at a point out along an infinite cable in terms of the voltage at the home end. In this $\beta = \sqrt{rcp}$. When a unit voltage is suddenly applied to an infinite cable, the problem is to evaluate

$$\epsilon^{-\sqrt{\gamma p}}\,1 \tag{507}$$

where $\gamma = rcx^2$.

Heaviside treated this by a series which is reviewed in Chapter XII. Jeffreys[1] evaluates it by a contour integral. Carson[2] first evaluated

$$pe^{-\sqrt{\gamma p}}\,1$$

[1] *Loc. cit.*, p. 23. The treatment involves the handling of a function with branch points, and hence we have not reviewed it in Chap. IX.

[2] Electric Circuit Theory and the Operational Calculus, *loc. cit.*, p. 71.

by noting that there is an infinite integral of the form[1]

$$\epsilon^{-\sqrt{\gamma p}} = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}\int_0^\infty \epsilon^{-a\lambda}\frac{\epsilon^{-\frac{\gamma}{4\lambda}}}{\lambda^{\frac{3}{2}}}\,d\lambda \tag{508}$$

so that by comparison,

$$p\epsilon^{-\sqrt{\gamma p}}\,1 = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}\,t^{-\frac{3}{2}}\,\epsilon^{-\frac{\gamma}{4t}} \tag{509}$$

and therefore

$$\begin{aligned}\epsilon^{-\sqrt{\gamma p}}\,1 &= \frac{1}{p}\left(p\epsilon^{-\gamma^{\frac{1}{2}}p^{\frac{1}{2}}}\right)1\\ &= \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}\int_0^t \lambda^{-\frac{3}{2}}\epsilon^{-\frac{\gamma}{4\lambda}}\,d\lambda.\end{aligned} \tag{510}$$

This is one form of solution. Chapter XII, Sec. 13, shows how it may be put in other forms which are more convenient for computation.

9. **Fractional Derivative with General Operands.** An expression for the half-order derivative when the operand is a general time function can be obtained by the use of Borel's theorem. Consider

$$p^{-\frac{1}{2}}f(t)1 = p^{-\frac{1}{2}}\frac{1}{Z(p)}\,1 = \frac{1}{p}\,p^{\frac{1}{2}}\frac{1}{Z(p)}\,1. \tag{511}$$

The last form is in shape for the use of the theorem, and thus

$$p^{-\frac{1}{2}}f(t)1 = \frac{1}{\sqrt{\pi}}\int_0^t \frac{f(\lambda)}{\sqrt{t-\lambda}}\,d\lambda.$$

From this,

$$p^{\frac{1}{2}}f(t)1 = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{f(\lambda)}{\sqrt{t-\lambda}}\,d\lambda, \tag{512}$$

subject of course to the comments already made regarding the derivative of a discontinuous function.

[1] Bierens de Haan, *loc. cit.*, p. 144, (16).

Equation (456) affords a means of expressing the fractional derivative or integral of a general operand $f(t)1$, for if this function can be expanded in a power series $\sum_0^\infty a_n t^n$, then $p^m f(t)1$ can be evaluated by the application of (456) to each term of the series. This can be expressed in the formula

$$p^m f(t)1 = \sum_0^\infty a_n \frac{\Gamma(n+1)t^{n-m}}{\Gamma(n-m+1)} 1 \tag{513}$$

where the restrictions on m and n again are that the gamma functions shall exist.

PROBLEMS ON CHAPTER XI

Prob. 1–11. Set up the infinite integral equation for the following operational formula:

$$p^{-n} 1 = \frac{t^n}{n!}$$

and by change of variable and the use of gamma functions, prove that the formula is correct.

Prob. 2–11. Evaluate $\Gamma(3.14)$, noting that $\Gamma(1.14)$, from a table of gamma functions, has the value 0.9365.

Prob. 3–11. Evaluate the operational expression

$$\frac{p}{\sqrt{p+a}} t^{\frac{3}{2}} 1.$$

Prob. 4–11. Two extensive blocks of insulating material having different thermal characteristics are separated by a thin metal heater plate. The heater current is suddenly turned on and so regulated over a period of 15 minutes as to maintain the input to the heater plate constant. Each portion of the heater plate one centimeter square has a total input of 0.05 watt. The heat capacity of the plate itself is negligible so that all of this input can be considered as immediately transferred to the insulating materials. Simultaneous readings taken at the end of this period on imbedded temperature measuring devices in the two materials indicate that the temperature gradients in the faces having intimate contact with the heater plate are as given below. The known constants of the materials are also given.

	Material A	*Material B*
gradient	10.8° C. per cm.	13.9° C. per cm.
specific gravity	0.9	1.2
specific heat	0.6	0.5

Assuming that the dimensions are great enough so that the heat flows substantially perpendicular to the heater surface in the region where the measuring devices are located, compute the thermal conductivities of the two materials.

Prob. 5–11. Verify the operational formula

$$\frac{p}{\sqrt{p^2 + a^2}}\left[\frac{\sqrt{p^2 + a^2} - p}{a}\right]^n 1 = J_n(at)$$

by means of the infinite integral theorem.

Prob. 6–11. Derive from the differential equations the time equation of the voltage which must be applied at the home end of an infinite cable with r and c to secure an entering current of constant value I.

Prob. 7–11. An alternating e.m.f. of rectangular wave form is suddenly applied to an infinite cable having only r and c, and the form of the entering current is desired. Write the operational expression that gives this response. Find the response by use of superposition.

Prob. 8-11 By use of the superposition theorem write an expression for the value of $p^{\frac{1}{2}} f(t)\ 1$, that is, an expression which may be utilized to evaluate half-order derivatives of general operands.

Using this expression, evaluate the half-order derivative of $\sin \omega t\ 1$.

Prob. 9–11. A steady current is suddenly impressed on an infinite cable. How does the terminal voltage vary? What is an analogous problem in the flow of heat?

Prob. 10–11. In Prob. 16–2 was considered the dissolving of a crystal. How will the concentration of the solution vary in the vicinity of the surface and how rapidly will the material go into solution?

CHAPTER XII

SERIES EXPANSIONS OF OPERATORS

Heaviside usually evaluated his operators by expanding them in series and applying them term by term to the unit function. This process has its distinct advantages. In the first place, if the legitimacy of the steps taken are not inquired into too diligently, only the simplest sort of mathematics is involved. Also it is a very powerful scheme, and results can be obtained in this way that are very hard to obtain otherwise. Then too, it is often possible to arrive at two series by expanding in different ways, and sometimes these are very convenient for computation — one for large and the other for small values of the time. The disadvantages are just as noteworthy. A series in powers of t may be easy to compute from, although sometimes it is not; but in any case it does not present a picture of the phenomenon which shows the influence of various factors, the limiting conditions, and so on, in the way a more compact expression does. We can look at the expression:

$$\sin(a + bt)$$

and tell a great deal about what it means in detail; but the corresponding series,

$$(a + bt) - \frac{1}{3!}(a + bt)^3 + \frac{1}{5!}(a + bt)^5 - \cdots$$

is rather meaningless unless it is recognized. It might be in this case, but if it appeared thus:

$$\begin{aligned} &a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \cdots \\ &+ bt\left(1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \cdots\right) \\ &- b^2t^2\left(\frac{1}{2}a - \frac{1}{12}a^3 + \cdots\right) + \cdots \end{aligned}$$

we should probably have difficulty in unscrambling or understanding it. The first disadvantage with the series expansions is the difficulty of recognizing or simplifying the result. The second disadvantage lies in the nature of the mathematics required to assure the validity of the results.

Heaviside depended upon experience and an innate grasp of the physical relationships to guide him away from mathematical pitfalls. It is remarkable that he slipped as seldom as he did; but in dealing with series developments of operators he sometimes went wrong.

Since the time of Heaviside the operational methods of analysis have been placed on a comparatively firm footing and have been increased in power. The series development has been the last to receive complete and satisfying treatment, and even to-day it is easier to raise questions in regard to it than to settle them. The work of Giorgi, Wiener, Carson, Stachó, and others has helped greatly. The first two of these will be followed principally, although usually only by presenting results and engineering rules rather than complete treatment.

2. **Series Expansions Yielding Convergent Results.** First consider the simplest kind of series development, where the result comes out in an absolutely convergent series, and where no terms such as $p^n 1$, with n a positive integer, are met with. Such a case appeared in

$$\frac{1}{p+b}1 = (p+b)^{-1}1 = (p^{-1} - p^{-2}b + p^{-3}b^2 - \cdots)1$$
$$= t - \frac{bt^2}{2!} + \frac{b^2t^3}{3!} - \cdots = \frac{1}{b}(1 - \epsilon^{-bt}). \qquad (514)$$

Now this result is of course correct, as is evident from other considerations; and this justifies the process in this case. One way of being sure of the steps is as follows:

If an operational expression $f(p)1$ is to be evaluated, and if there can be found, no matter how, an expression for

$A(t)$ which, inserted in the integral:

$$\int_0^\infty \epsilon^{-a\lambda} A(\lambda)\, d\lambda$$

will yield on integration (a being any positive real) an expression which is identically equal to $\frac{1}{a} f(a)$, then the operational expression has been correctly evaluated.

Thus, if $f(p)$ can be expanded in a series, applied term by term to the unit function to obtain a convergent series for A, and this upon being integrated as above, gives again an expression which is truly equal to $\frac{f(a)}{a}$, where a is a positive real, we have a correct process.

Now Volterra, to whom we owe much of the mathematical treatment of operational processes, has given[1] a criterion which justifies expansions which lead to convergent results. This theorem deals with more general operational processes than we are here concerned with, but in terms of our special type of operators and operand it may be expressed thus:

If the series

$$a_1 u + a_2 u^2 + a_3 u^3 + \cdots$$

is convergent for any value of u, less in absolute magnitude than some assignable constant, then the result of

$$\left(\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots\right) 1$$

obtained by the use of equation (463), or by treating p^{-1} as $\int_0^t dt$, will be a convergent series in t which correctly expresses the result of the operation from which this power series is developed.

[1] Leçons sur les Équations Intégrales, Gauthier-Villars, Paris, 1913, p. 153. Rice Institute Pamphlet, Vol. 7, No. 4, October, 1920.

3. **An Example of a Convergent Development.** An important operator, as will be shown, is

$$\epsilon^{b(p-\sqrt{p^2-\sigma^2})}$$

and another is

$$\epsilon^{b(p-\sqrt{p^2-\sigma^2})}\frac{p}{\sqrt{p^2-\sigma^2}}.$$

These are not simple and unfortunately their treatment is necessarily rather involved, but they can be handled by series that lead to convergent results. The first one may be manipulated as follows: Expand the radical by the binomial theorem:

$$(p^2-\sigma^2)^{\frac{1}{2}} = p - \frac{1}{2}p^{-1}\sigma^2 - \frac{1}{2^2}\frac{1}{2!}p^{-3}\sigma^4 - \cdots. \quad (515)$$

Insert this, and write the operator as an infinite product:

$$\epsilon^{b(p-\sqrt{p^2-\sigma^2})} = \left(\epsilon^{\frac{b\sigma^2}{2p}}\epsilon^{\frac{b\sigma^4}{2^2 2!\,p^3}}\epsilon^{\frac{3\,b\sigma^6}{2^3 3!\,p^5}}\cdots\right). \quad (516)$$

Now expand each of these terms:

$$\left(1+\frac{b\sigma^2}{2p}+\frac{b^2\sigma^4}{2^2 2!\,p^2}+\cdots\right)\left(1+\frac{b\sigma^4}{2^2 2!\,p^3}+\frac{b^2\sigma^8}{2^4 2!^3\,p^6}+\cdots\right)$$
$$(\cdots)(\cdots).$$

Multiply these together, collect terms of successive powers of p, and apply to the unit function:

$$\left\{1+\frac{b\sigma^2}{2}p^{-1}+\frac{b^2\sigma^4}{2^2 2!}p^{-2}+\left(\frac{b^3\sigma^6}{2^3 3!}+\frac{b\sigma^4}{2^2 2!}\right)p^{-3}\right.$$
$$\left.+\left(\frac{b^4\sigma^8}{2^4 4!}+\frac{b^2\sigma^6}{2^3 2!}\right)p^{-4}+\cdots\right\}1.$$

Only integrations are now involved, and the result is

$$\epsilon^{b(p-\sqrt{p^2-\sigma^2})}\,1 = 1+\frac{b\sigma^2 t}{2}+\frac{b^2\sigma^4 t^2}{2^2 2!^2}+\left(\frac{b^3\sigma^6}{2^3 3!^2}+\frac{b\sigma^4}{2^2 2! 3!}\right)t^3$$
$$+\left(\frac{b^4\sigma^8}{2^4 4!^2}+\frac{b^2\sigma^6}{2^3 2!\,4!}\right)t^4+\cdots. \quad (517)$$

The series:

$$\left(1 + \frac{b\sigma^2}{2}u + \frac{b^2\sigma^4}{2^2 2!}u^2 + \cdots\right)$$

formed from the above is in fact convergent for any value of u, so the Volterra condition is satisfied, and the convergent result is correct.

Another way of obtaining the same result is to expand the exponential first, and the radicals afterwards. It does not appear possible to express the result in more compact form. It may be checked by working backwards through the integral equation.

Now, to evaluate the second expression, consider

$$\frac{p}{\sqrt{p^2 - \sigma^2}}$$

which may be expanded into

$$1 + \frac{\sigma^2}{2}p^{-2} + \frac{3}{2^2 2!}\sigma^4 p^{-4} + \cdots.$$

Again only integrations are involved, and when this expression is applied term by term to expression (517) and equal powers of t are gathered together, there results

$$1 + \frac{b\sigma^2 t}{2} + \left(\frac{b^2\sigma^4}{2^2 2!^2} + \frac{\sigma^2}{2 \cdot 2!}\right)t^2 + \left(\frac{b^3\sigma^6}{2^3 3!^2} + \frac{b\sigma^4}{2^2 2!^2}\right)t^3 + \left(\frac{b^4\sigma^8}{2^4 4!^2} + \frac{b^2\sigma^6}{2^5 2!^3} + \frac{\sigma^4}{2^4 2!^2}\right)t^4 + \cdots. \tag{518}$$

This certainly looks bad enough. However, it will simplify later.

4. **Traveling Waves on an Infinite Transmission Line.** Now let us see why we are interested in such inconvenient operators. The differential equations of a transmission line with all four parameters present are

$$\left.\begin{aligned}(r + lp)i &= -\frac{\partial e}{\partial x}\\ (g + cp)e &= -\frac{\partial i}{\partial x}\end{aligned}\right\}. \tag{519a}$$

Treating these equations in the same way as the differential equations for the cable of Sec. 1, Chapter XI, the same operational solutions

$$\left.\begin{aligned} e &= \epsilon^{-\beta x} e_0 \\ i &= \epsilon^{-\beta x} i_0 \end{aligned}\right\} \tag{519b}$$

are obtained, where now

$$\beta = \frac{1}{v}\sqrt{(p+\rho)^2 - \sigma^2} \tag{520}$$

and

$$\left.\begin{aligned} \rho &= \frac{r}{2l} + \frac{g}{2c} \\ \sigma &= \frac{r}{2l} - \frac{g}{2c} \\ v &= \frac{1}{\sqrt{lc}} \end{aligned}\right\}. \tag{521}$$

These same expressions can also be obtained by writing the alternating-current formulas and then replacing $j\omega$ by p. The home-end current for a suddenly-applied e.m.f. has already been treated in Chapter XI and found to be

$$i_0 = E_0\sqrt{\frac{c}{l}}\,\epsilon^{-\rho t} J_0(j\sigma t) + E_0 g v \int_0^t \epsilon^{-\rho t} J_0(j\sigma t)\, dt. \tag{522}$$

The much more difficult problem of the effects at a distance x out on the line now arises. The voltage at a distance x is given by

$$e = E_0 \epsilon^{-\frac{x}{v}\sqrt{(p+\rho)^2 - \sigma^2}}\, 1. \tag{523}$$

First a damping factor can be removed:

$$e = E_0 \epsilon^{-\rho t} \frac{p}{p-\rho}\, \epsilon^{-\frac{x}{v}\sqrt{p^2 - \sigma^2}}\, 1. \tag{524}$$

Now it is necessary to evaluate

$$\epsilon^{-\frac{x}{v}\sqrt{p^2 - \sigma^2}}\, 1.$$

This must represent a function which is zero until the time x/v, for nothing can happen at a distance x out along the line until a sufficient time has elapsed for a wave to travel from the home end to the point in question. This suggests that we transfer the time axis, which can be done by employing $\epsilon^{\frac{xp}{v}}$. Thus

$$\epsilon^{-\frac{xp}{v}}\left(\epsilon^{\frac{xp}{v}}\epsilon^{-\frac{x}{v}\sqrt{p^2-\sigma^2}}\right)1$$

$$= \epsilon^{-\frac{xp}{v}}\left(\epsilon^{\frac{x}{v}(p-\sqrt{p^2-\sigma^2})}\right)1. \tag{525}$$

We have now arrived at something we recognize. Inserting the series (517) there results for the entire expression:

$$e = E_0\epsilon^{-\rho t}\epsilon^{-\frac{xp}{v}}\frac{p}{p-\rho}\left(1+\frac{x\sigma^2 t}{2v}+\frac{x^2\sigma^4 t^2}{2^2 2!^2 v^2}+\cdots\right)1. \tag{526}$$

Write the fractional operator $\dfrac{p}{p-\rho}$ as

$$(1+\rho p^{-1}+\rho^2 p^{-2}+\cdots)$$

and apply, obtaining

$$e = E_0\epsilon^{-\rho t}\epsilon^{-\frac{xp}{v}}\left[1+\left(\frac{x\sigma^2}{2v}+\rho\right)t+\left(\frac{x^2\sigma^4}{2^2 2!^2 v^2}+\frac{x\sigma^2\rho}{2^2 v}+\frac{\rho^2}{2!}\right)t^2+\cdots\right]1. \tag{527}$$

Now it is possible to use

$$\epsilon^{-\frac{xp}{v}}f(t)\,1 = f\left(t-\frac{x}{v}\right)1_{\frac{x}{v}} \tag{528}$$

which gives the voltage:

$$e = E_0\epsilon^{-\rho t}\left[1+\left(\frac{x\sigma^2}{2v}+\rho\right)\left(t-\frac{x}{v}\right)+\left(\frac{x^2\sigma^4}{2^2 2!^2 v^2}+\frac{x\sigma^2\rho}{2^2 v}+\frac{\rho^2}{2!}\right)\left(t-\frac{x}{v}\right)^2+\cdots\right]1_{\frac{x}{v}}. \tag{529}$$

This is useful for computation only when $\left(t - \frac{x}{v}\right)$ is small. As Heaviside remarked, it is unfortunate that the result should be so complicated, but there is apparently no way of writing it more simply in terms of any functions which have been named and catalogued.

We shall have better success with the current. At first only the first term of (522) will be treated, for the other can then be obtained by an integration. Now the expression to be evaluated is

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\beta x}\epsilon^{-\rho t}J_0(j\sigma t) = E_0\sqrt{\frac{c}{l}}\epsilon^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}\epsilon^{-\rho t}J_0(j\sigma t). \tag{530}$$

The exponential can be shifted, giving

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\epsilon^{-\frac{x}{v}\sqrt{p^2-\sigma^2}}J_0(j\sigma t). \tag{531}$$

Then the Bessel function can be put in operational form, as in equation (474), giving

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\epsilon^{-\frac{x}{v}\sqrt{p^2-\sigma^2}}\frac{p}{\sqrt{p^2-\sigma^2}}1 \tag{532}$$

and the time axis transferred as before:

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\epsilon^{-\frac{xp}{v}}\epsilon^{\frac{x}{v}(p-\sqrt{p^2-\sigma^2})}\frac{p}{\sqrt{p^2-\sigma^2}}1. \tag{533}$$

Again a recognizable operator is encountered, the evaluation of which has already been made in (518), thus

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\epsilon^{-\frac{xp}{v}}\left[1 + \left(\frac{x\sigma^2}{2\,v}\right)t + \left(\frac{x^2\sigma^4}{2^2 2!^2 v^2} + \frac{\sigma^2}{2\cdot 2!}\right)t^2 + \left(\frac{x^3\sigma^6}{2^3 3!^2 v^3} + \frac{x\sigma^4}{2^2 2!^2 v}\right)t^3 + \cdots\right]1. \tag{534}$$

Now apply $\epsilon^{-\frac{xp}{v}}$ and this becomes

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\left[1+\left(\frac{x\sigma^2}{2v}\right)\left(t-\frac{x}{v}\right)\right.$$
$$+\left(\frac{x^2\sigma^4}{2^2 2!^2 v^2}+\frac{\sigma^2}{2\cdot 2!}\right)\left(t-\frac{x}{v}\right)^2$$
$$\left.+\left(\frac{x^3\sigma^6}{2^3 3!^2 v^3}+\frac{x\sigma^4}{2^2 2!^2 v}\right)\left(t-\frac{x}{v}\right)^3+\cdots\right]1_{\frac{x}{v}} \quad (535)$$

By collecting terms in the order of increasing powers of σ, this may be written

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\left[1+\frac{\sigma^2}{2^2}\left(t^2-\frac{x^2}{v^2}\right)\right.$$
$$\left.+\frac{\sigma^4}{2^2 4^2}\left(t^4-\frac{2x^2t^2}{v^2}+\frac{x^4}{v^4}\right)+\cdots\right]1_{\frac{x}{v}} \quad (536)$$

and the quantity in brackets is recognized as the expansion of

$$J_0\left[j\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]$$

and so, finally,

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}J_0\left[j\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]1_{\frac{x}{v}} \quad (537)$$

This is the first term of the arrival current at the point x, and the complete expression is

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}J_0\left[j\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]1_{\frac{x}{v}}$$
$$+E_0 gv\int_0^t \epsilon^{-\rho t}J_0\left[j\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]1_{\frac{x}{v}}\,dt \quad (538)$$

which can be checked again by insertion in the infinite integral.

5. **The Finite Line.** While this result is being dealt with consider the case of a finite line, of length D. The result

above for an infinite line is expressed in terms of waves. The current at any point out on the line is zero for a time $t = \frac{x}{v}$ after the switch is closed. At this instant the wave from the home end arrives and the current jumps abruptly to a value $E_0\sqrt{\frac{c}{l}}\epsilon^{-\frac{\rho x}{v}}$ and thereafter changes slowly. This first part of the phenomenon at x would evidently proceed in exactly the same manner if the line were terminated at a point further out than x along the line; for, naturally, conditions at a terminal cannot affect the wave until the wave arrives. So, if we can find what happens to the wave at the line terminals, the solution above can be used for a finite line also. The simplest cases are those in which the distant end of the line is either open or grounded. Something will be said regarding other cases later.

At the open end of a line the current is forced to remain zero. When a wave of current arrives at an open end, therefore, it is reflected in such a way as to maintain the current at the end zero. This it can do only if the current wave is completely reflected with a changed sign so that the outgoing and returning waves cancel.

It arrives again at the point x after a total time $t = \frac{2D - x}{v}$. When the reflected wave arrives at the home end, it is again reflected. But this time, as can be readily shown, the reflection is without a change of sign; for the home end is grounded as far as impedance to the wave is concerned. Thus the complete expression for a finite open line becomes

$$i = E_0\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\left\{J_0\left[j\sigma\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]I_{\frac{x}{v}} - J_0\left[j\sigma\left(t^2 - \frac{(2D - x)^2}{v^2}\right)^{\frac{1}{2}}\right]I_{\frac{2D-x}{v}} - J_0\left[j\sigma\left(t^2 - \frac{(2D + x)^2}{v^2}\right)^{\frac{1}{2}}\right]I_{\frac{2D+x}{v}} - \cdots\right\} \quad (539)$$

where, for simplicity, only the series corresponding to the first term of the current expression is written, it being understood that there follows a corresponding series of integrals as a result of the second term of the current expression. Expression (539) can of course also be used for evaluating the current at the home end of a finite line by inserting $x = 0$,

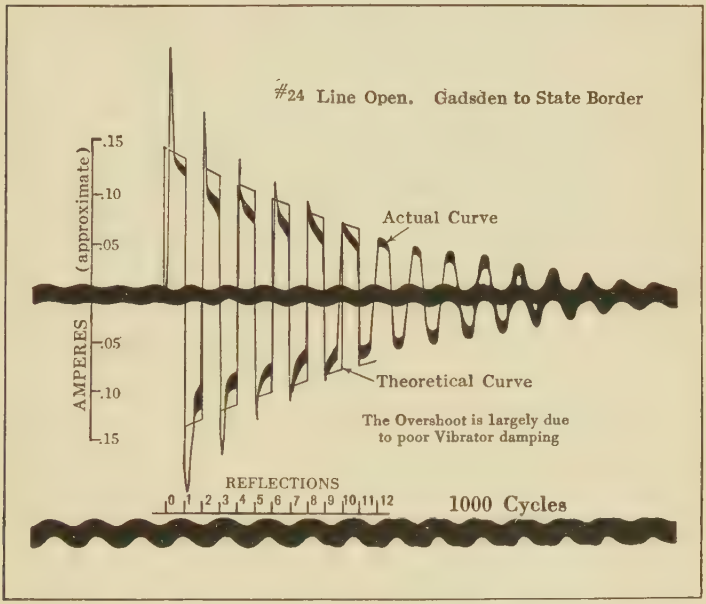

FIG. 44. Oscillogram on an Actual Aerial Line. Entering Current with Distant End Open. (Trans. A.I.E.E. 1923, p. 882.)

in which case the terms combine in pairs. Figure 44 shows the calculated current curve as well as the actual oscillogram for a transmission line open at the distant end. If the distant end of the line is grounded, the expression is identical with the above, but with the signs of all terms plus. This is shown for an actual line in Fig. 45, and for an artificial line in Fig. 46.

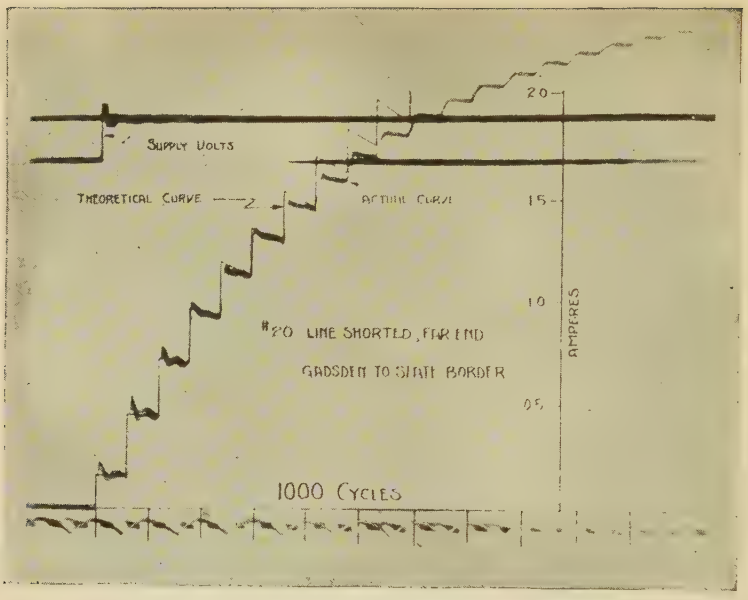

FIG. 45. Oscillogram on an Actual Aerial Line. Entering Current with Distant End Short Circuited. (Trans. A.I.E.E. 1923, p. 882.)

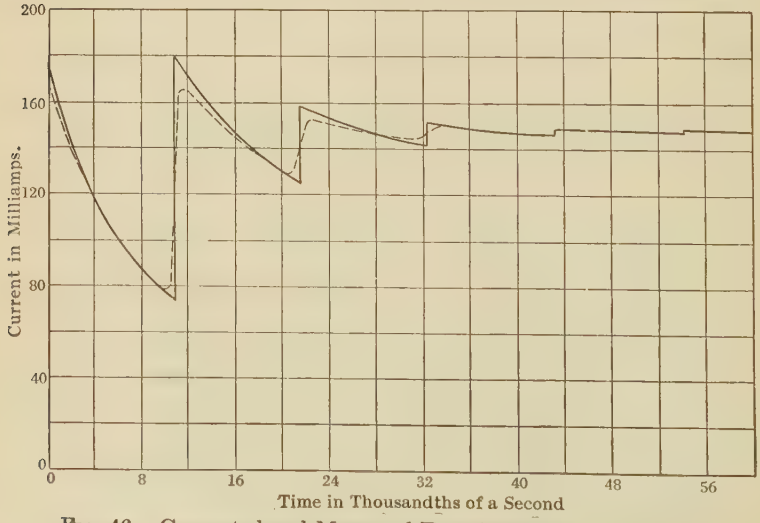

FIG. 46. Computed and Measured Transients of an Artificial Line. (Trans. A.I.E.E. 1923, p. 881.)

A great deal has been written[1] on the subject of traveling waves on transmission lines, but the matter will be considered here only in connection with operational processes of analysis.

6. **Another Convergent Result.** In Chapter XI, Sec. 7, a convergent series was developed by a process of successive integrations by parts for the operational expression:

$$\frac{\sqrt{p}}{\sqrt{p}+\sqrt{b}}\,1.$$

This same series can be obtained directly with much greater ease. By dividing, the operator can be readily expanded into

$$(1 - b^{\frac{1}{2}}p^{-\frac{1}{2}} + bp^{-1} - b^{\frac{3}{2}}p^{-\frac{3}{2}} + \cdots)1$$

leaving for interpretation:

$$\left(1+\frac{b}{p}+\frac{b^2}{p^2}+\cdots\right)1$$
$$- b^{-\frac{1}{2}}\left(\frac{b}{p}+\frac{b^2}{p^2}+\cdots\right)p^{\frac{1}{2}}\,1.$$

Using (463), or interpreting $p^{\frac{1}{2}}\,1$ as $1/\sqrt{\pi t}$ and performing the indicated integrations, there is obtained

$$\frac{\sqrt{p}}{\sqrt{p}+\sqrt{b}}\,1 = \left(1 + bt + \frac{b^2t^2}{2!} + \cdots\right)$$
$$- \frac{1}{\sqrt{\pi bt}}\left(2\,bt + \frac{(2\,bt)^2}{1\cdot 3} + \frac{(2\,bt)^3}{1\cdot 3\cdot 5} + \cdots\right) \quad (540)$$

[1] For example, K. W. Wagner, Elektromagnetische Ausgleichvorgänge in Freileitungen und Kabeln, Teubner, Leipzig, 1908.

Kennelly and Nabeshima, The Transient Process of Establishing a Steadily Alternating Current on a Long Line, Proc. Am. Phil. Soc., **59**, 1920, p. 325.

R. Rüdenberg, Elektrische Schaltvorgänge, Springer, Berlin, 1926.

V. Bush, Transmission Line Transients, Trans. A.I.E.E., **42**, 1923, p. 878. Transmission Line Transient Investigations, M.I.T., E.E. Research Bulletin, No. 50, 1925, p. 12.

J. H. Cox, Transmission Line Voltage Surges, Trans. A.I.E.E., **46**, 1927, p. 330.

which is exactly what we had before. This illustrates the ease with which such a development can often be obtained.

7. **Divergent Series. A Simple Example.** The series obtained for results so far have all been convergent. These are in general convenient for computation only when t is small; for when t is large, many terms need to be taken into account. It is often possible and very convenient to obtain an alternative form of series, which, while it is divergent, can be used for computation when t is large. In the useful forms of such series the error involved when they are employed for computation is less than the value of the last term included. One form of divergent series — the asymptotic series — often has this property when used to compute the results of physical problems. The question of when and how divergent series may with propriety be developed involves advanced mathematical considerations. Fortunately there is a rigorous treatment of this matter, due to Wiener,[1] for the asymptotic series resulting from operators of the sort under consideration, and we shall follow his method. This gives criteria which govern the propriety of asymptotic developments. Examination of the possible error in utilizing

[1] N. Wiener, The Operational Calculus, Math. Annalen, Band 95, Heft 4. 1926. Giorgi also gives a treatment of this matter, but I prefer Wiener's treatment here, for one reason because the papers of Giovanni Giorgi are not available to many readers. The papers concerned are: Sul Calcolo delle Soluzioni Funzionali, Associazione Elettrotecnica Italiana Atti, Vol. IX, 1905, p. 651; On the Functional Dependence of Physical Variables, Proc. of the International Congress of Mathematics, Toronto, 1924; Sugli Integrali dell'Equazione di Propagazione in una Dimensione, Rendiconti del Circolo Matematico di Palermo, **52**, 1928.

Carson, *loc. cit.*, also discusses the series solutions.

Stachó discusses the conditions for an asymptotic expansion from a somewhat different viewpoint. Tibor v Stachó, Operationskalkül von Heaviside und Laplacesche Transformation; Acta Littera ac Scientiarum, Regiae Universitatis Hungaricae Francisco-Josephinae, Math. Section, Tom III, Fasc. II–III, 1927. His treatment is more limited in certain respects than that of Wiener.

an asymptotic series for the computation of a result of a physical problem where the series has been developed as a result of operational methods will require further study.

First, however, let us consider an example, to fix ideas. In Sec. 4, Chapter XI, we treated the operational formula:

$$\frac{p}{\sqrt{p^2 + 2bp}}\,1 = \epsilon^{-bt} J_0(jbt) \tag{541}$$

the result being written here in the form of a convergent series. It is desirable to obtain the alternative form. Expand the operator by the binomial theorem:

$$p(2bp+p^2)^{-\frac{1}{2}}\,1 = \left[\frac{1}{(2b)^{\frac{1}{2}}}p^{\frac{1}{2}} - \frac{1}{2(2b)^{\frac{3}{2}}}p^{\frac{3}{2}} + \frac{1\cdot 3}{2^2 2!}\frac{1}{(2b)^{\frac{5}{2}}}p^{\frac{5}{2}} - \cdots\right]1. \tag{542}$$

Using equation (463), there results

$$\frac{p}{\sqrt{p^2 + 2bp}}\,1 = \frac{1}{\sqrt{\pi}}\left[\frac{1}{(2bt)^{\frac{1}{2}}} + \frac{1}{2^2(2bt)^{\frac{3}{2}}} + \frac{1^2\cdot 3^2}{2^4 2!\,(2bt)^{\frac{5}{2}}} + \cdots\right]$$

$$= \frac{1}{\sqrt{2\pi bt}}\left[1 + \frac{1}{8bt} + \frac{1^2 3^2}{2!\,(8bt)^2} + \cdots\right] \tag{543}$$

where the equals sign is used formally with the comment that there is a divergent series in the result. Several interesting points regarding this series may immediately be noted. For any given value of t the successive terms at first become smaller; but as we proceed out along the series we reach a smallest term, and after that the terms increase and ultimately become very large. Yet this is a convenient series. The result of our operational expression or the value of $\epsilon^{-bt}J_0(jbt)$ can be computed with ease when t is large. Suppose, for example, $8bt = 100$. The second term in brackets is 0.01 and the third, 0.00045. Two terms will give a precision of at least one per cent; while three will give at least 0.045 per cent. For values of $8bt$ greater than 100, and for a precision of one per cent or better, it is permissible to use simply $(2\pi bt)^{-\frac{1}{2}}$. The larger the value of t, the fewer terms

need be used to obtain a given precision, and the higher the precision that can be obtained. The larger the value of t the farther from the beginning of the series will be found the smallest term, beyond which it is useless to proceed. Evidently a series of this sort may be quite useful. So it is important to be able to obtain such asymptotic series, and to be sure of the possibilities and limitations of results in this form.

8. **Asymptotic Series.** An asymptotic series expansion for a function $f(t)$ is a series the partial sums of which approximate the function at large values of the argument t. It may be an expansion in descending integral powers of t, in which case it has the form

$$b_0 + \frac{b_1}{t} + \frac{b_2}{t^2} + \frac{b_3}{t^3} + \cdots.$$

The difference between the function and $k + 1$ terms of the series becomes small compared with $(1/t)^k$ as t increases without limit. This means that if the kth partial sum is given by

$$S_k(t) = b_0 + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_{k-1}}{t^{k-1}} + \frac{b_k}{t^k} = \sum_{n=0}^{n=k} b_n t^{-n}$$

then

$$\lim_{|t|\to\infty} t^k[f(t) - S_k(t)] = 0 \text{ for fixed } k.$$

This can also be expressed as

$$\lim_{|t|\to\infty} t^k\left[f(t) - \sum_{n=0}^{n=k} b_n t^{-n}\right] = 0 \text{ for fixed } k.$$

This expression enables us to obtain the coefficients b_n of the asymptotic expansion of $f(t)$.

Thus

$$\left.\begin{aligned} &\lim_{t\to\infty} f(t) = b_0 \\ &\lim_{t\to\infty} t[f(t) - b_0] = b_1 \\ &\lim_{t\to\infty} t^2\left[f(t) - b_0 - \frac{b_1}{t}\right] = b_2 \text{ etc.} \end{aligned}\right\} \tag{544}$$

An analytic function of t can be expanded in the neighborhood of a finite point t_0 in terms of a Taylor's series — that is, in terms of the value of the function and its derivatives at $t = t_0$. Such an expansion completely determines the function for values of t near t_0.

An asymptotic series may be regarded as an expansion about the "point at infinity." It expresses in detail the manner in which the function behaves as the argument becomes infinite. It does not, however, completely determine the function; for two functions which are altogether different in the finite range may approach infinity in identical manner, and may hence be represented for large values of the argument by the same asymptotic series.

In expanding an analytic function about a finite point by Taylor's series, we write down as partial sums, in adding terms successively, the nearest straight line, the nearest parabola, the nearest cubic, and so on. By "nearest" is meant with the highest order of contact with the function at the point. Thus the expansion of the exponential about $t = 0$,

$$\epsilon^{bt} = 1 + bt + \frac{b^2t^2}{2!} + \cdots$$

may be plotted as in Fig. 47. The error made in stopping at the $k + 1$ term approaches zero faster than t^k when t approaches zero.

An asymptotic expansion may be considered to be a similar sort of device, except that the point about which the expansion is desired is removed to infinity. The first term b_0 is the constant value which the function to be represented approaches. This with the next term gives the nearest hyperbola of the form $b_0 + b_1/t$ that can be chosen. The next term gives the nearest cubic curve $b_0 + b_1/t + b_2/t^2$, and so on. If $k + 1$ terms like these are included in the series it will fit the function, as t becomes very large, with an error not greater than $1/t^k$. The process appears clear if the

plane on which our curves are plotted is projected or foreshortened so that the point at ∞ is transformed into a finite

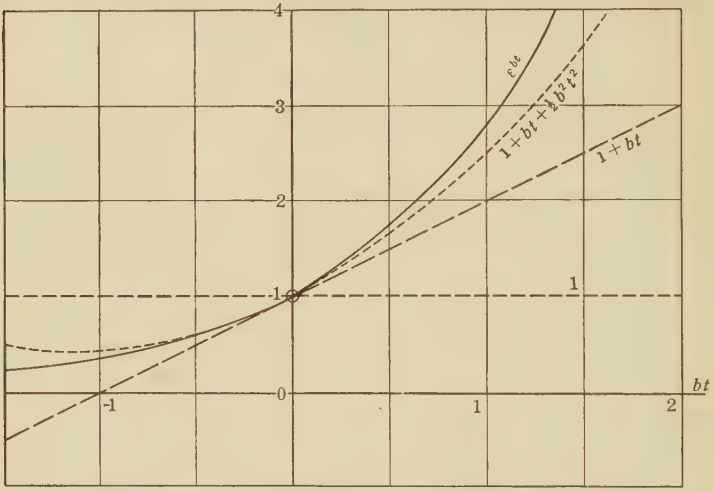

Fig. 47. Successive Approximations to an Exponential.

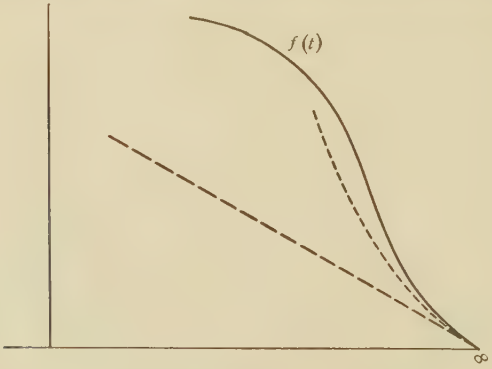

Fig. 48.

point. This may be accomplished by a change of variable of the sort $t_1 = \dfrac{t}{1+t}$, for example. Then, as in Fig. 48, the series plotted for a successively greater number of terms will

be seen to fit more and more closely to the function as the point representing ∞ is approached.

A function which behaves exponentially at infinity, such as ϵ^{-bt}, cannot be usefully represented by an asymptotic series, for it becomes zero at $t = \infty$ to a higher order than t^{-k}, no matter what value we pick for k. No term in the form of a simple power of t can thus serve as a good approximation for an exponential when t is large enough. So the asymptotic expansion of ϵ^{-bt} is simply a string of zeros. This is easily checked by evaluating the coefficients of this expansion of ϵ^{-bt} by (544).

Thus

$$\lim_{t\to\infty} \epsilon^{-bt} = 0$$
$$\lim_{t\to\infty} t[\epsilon^{-bt} - 0] = 0$$
$$\lim_{t\to\infty} t^2[\epsilon^{-bt} - 0 - 0] = 0.$$

It is interesting to compare this with the power series expansion of the function ϵ^{-1/x^2} about $x = 0$. This function is zero when $x = 0$, and moreover all of its derivatives are zero when $x = 0$, as may readily be shown by writing several of them out. Hence every term in the Maclaurin expansion of this particular function is zero, and the expansion itself is a string of zeros. Now suppose we have some usual function $f(x)$ and its Maclaurin expansion:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \qquad (545)$$

and that we write the formal Maclaurin expansion for $f(x) + \epsilon^{-1/x^2}$. Evidently exactly the same result as before will be obtained. To state the matter in a somewhat different way: Suppose a problem has been solved and a result obtained as a series:

$$bx - \frac{b^3x^3}{3!} + \frac{b^5x^5}{5!} - \cdots.$$

We should recognize this series and be inclined to write our result as $\sin bx$. But it is to be noted that the partial sums of the series have the same order of contact with

$$\sin bx + k\epsilon^{-\frac{1}{x^2}}$$

at $x = 0$ as with $\sin bx$. The series is hence the formal power series expansion for either function. Of course in any practical problem such a matter is usually settled by regarding the reasonableness of the form of the result, and terms such as $\epsilon^{-\frac{1}{x^2}}$ rarely belong to a true solution of physical problems. Generally when obtaining a convergent series in arriving at a solution to a physical problem, there is obtained at the same time an assurance that it converges to the solution of the problem set. Such an assurance is usually lacking with asymptotic series, for operational methods generally produce asymptotic series which do not converge.

It follows from the above that any function, and the same function plus a term which vanishes exponentially at infinity, will have the same asymptotic expansion. Failure to realize this has sometimes caused confusion. Of course an operator can sometimes be split into the sum of two parts, one of which yields a result which vanishes exponentially at infinity, and the other of which is representable by an asymptotic series. Provided both parts are correctly interpreted in accordance with the same definitions, it is quite proper to retain both portions for computation, examining the error of the asymptotic series alone for any given value of t, and adding in the exponential if it is comparable in magnitude at this value of t. But, if t is taken large enough, the same result will be obtained whether the exponential is included or not, and to any desired precision. Negative exponents are considered, of course, since they alone can occur in dealing with dissipative networks.

It happens that when the unit function is dealt with as

operand, any rational algebraic operator yields a result which behaves exponentially at infinity. Therefore this result cannot be usefully put into the form of an asymptotic series. Hence our asymptotic developments are obtained from irrational operators applied to the unit operand; or, what amounts to much the same thing, rational operators applied to an operand of the form $t^{-\frac{1}{2}}\,1$. It is for this reason that the asymptotic series which are obtained by operational processes involve the fractional rather than the integral powers of t. They can be regarded as a product of $t^{\frac{1}{2}}$ and an asymptotic series in integral powers of t.

The terms of an asymptotic series may all have the same sign, or the signs may alternate. In the latter case the value of the series, as successive terms are added, oscillates about the value of the function represented. If an asymptotic series has terms of alternating sign, if the terms decrease in size before they begin to increase as the number of terms is increased, and if the position of the smallest term recedes as t is increased, then the series taken up to the smallest term has an order of contact with the function at infinity higher than any t^{-k}.

9. **Example of an Asymptotic Development.** It will be of interest to first develop in detail the asymptotic expansion of a particular operational expression in order to illustrate the general reasoning necessary to establish rigorously an asymptotic series. This example is chosen particularly to illustrate the fact that a function which vanishes exponentially at infinity contributes nothing to an asymptotic expansion. Of course it would be preferable to carry out such an analysis on a general class of operators rather than on a single operator, hence avoiding the details of particular cases. It would then be possible to obtain a particular development by first simply noting that the operator falls within the classification, and then following very simple rules. This is in fact the scheme that shall be generally adopted, for it avoids a great deal of detailed mathematics of a type which

is none too simple. The special example chosen, however, has interesting features of its own.

Consider the operational expression:

$$\frac{\sqrt{p}}{p+b}\,1$$

which may be written

$$p^{-\frac{1}{2}}\frac{p}{p+b}\,1 = p^{-\frac{1}{2}}\,\epsilon^{-bt}\,1. \tag{546}$$

Using the expression obtained for the fractional-order derivative of a function, we may write

$$\frac{\sqrt{p}}{p+b}\,1 = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{t-\lambda}}\,d\lambda \tag{547}$$

or by employing Borel's theorem in the alternative form,

$$\frac{\sqrt{p}}{p+b}\,1 = \frac{1}{\sqrt{\pi}}\,\epsilon^{-bt}\int_0^t \epsilon^{b\lambda}\lambda^{-\frac{1}{2}}\,d\lambda. \tag{548}$$

This expression is to be developed into an asymptotic series. First split into two parts:

$$\frac{\sqrt{p}}{p+b}\,1 = \frac{1}{\sqrt{\pi}}\,\epsilon^{-bt}\int_1^t \epsilon^{b\lambda}\lambda^{-\frac{1}{2}}\,d\lambda + \frac{1}{\sqrt{\pi}}\,\epsilon^{-bt}\int_0^1 \epsilon^{b\lambda}\lambda^{-\frac{1}{2}}\,d\lambda. \tag{549}$$

Now the second integral in this expression is equal to some finite constant. Its actual value is not needed. It is multiplied by a negative exponential. Hence the entire term vanishes exponentially at infinity. Its asymptotic expansion is hence simply a string of zeros, and it contributes nothing to the final result. We shall therefore cancel it and retain only the first term. Integrate this first integral by parts:

$$\frac{1}{\sqrt{\pi}}\,\epsilon^{-bt}\left[\frac{1}{b}\,\epsilon^{b\lambda}\lambda^{-\frac{1}{2}}\right]_1^t + \frac{\epsilon^{-bt}}{2\,b\,\sqrt{\pi}}\int_1^t \epsilon^{b\lambda}\lambda^{-\frac{3}{2}}\,d\lambda$$

$$= \frac{1}{b\,\sqrt{\pi}}\,t^{-\frac{1}{2}} - \frac{1}{b\,\sqrt{\pi}}\,\epsilon^{-bt}\epsilon^{b} + \frac{\epsilon^{-bt}}{2\,b\sqrt{\pi}}\int_1^t \epsilon^{b\lambda}\lambda^{-\frac{3}{2}}\,d\lambda. \tag{550}$$

Again it will be noticed that the second term vanishes exponentially and can be dropped. Another integration by parts can be performed and this process repeated time after time. This gives after n repetitions:

$$\frac{\sqrt{p}}{p+b}\,1 = \frac{1}{b\sqrt{\pi t}}\left(1+\frac{1}{2\,bt}+\frac{1\cdot 3}{(2\,bt)^2}+\cdots+\frac{1\cdot 3\cdots(2n-3)}{(2\,bt)^{n-1}}\right)$$
$$+\frac{1\cdot 3\cdots(2n-1)}{(2\,b)^n\sqrt{\pi}}\,\epsilon^{-bt}\int_1^t \epsilon^{b\lambda}\lambda^{-\frac{2n+1}{2}}\,d\lambda$$
$$+ \text{ a term vanishing exponentially.} \qquad (551)$$

We now have the process for obtaining the series; but before the validity of the process is assured, the value of the remainder which is the integral term in expression (551) must be estimated. If the balance of the expression is to develop into an asymptotic series, then this remainder should vanish at least as rapidly as the last term — that is, at least as rapidly as $t^{-n+\frac{1}{2}}$ when t becomes infinite. The constant is unimportant. The balance of this remainder term is written in two parts, thus:

$$\epsilon^{-bt}\int_1^{\frac{t}{2}} \epsilon^{b\lambda}\lambda^{-\frac{2n+1}{2}}\,d\lambda + \epsilon^{-bt}\int_{\frac{t}{2}}^{t} \epsilon^{b\lambda}\lambda^{-\frac{2n+1}{2}}\,d\lambda.$$

The value of the first integral cannot be greater than the product of $\epsilon^{\frac{bt}{2}}$ and $\frac{t}{2}$, as is best seen by plotting the functions under the integral. Hence the first term is at least as small as $\frac{t}{2}\,\epsilon^{-\frac{bt}{2}}$. But this vanishes exponentially at infinity and hence more rapidly than t^{-k} no matter how large k may be. The first part of the remainder thus behaves satisfactorily. The second integral, similarly, is certainly less than $\frac{t}{2}$ times the product of ϵ^{bt} and $\left(\frac{t}{2}\right)^{-\frac{2n+1}{2}}$. The entire

term is thus less than $\left(\frac{t}{2}\right)^{-\frac{2n-1}{2}}$. Thus the entire remainder satisfies the condition for an asymptotic development. We have then, asymptotically,

$$\frac{\sqrt{p}}{p+b}1 = \frac{1}{b\sqrt{\pi t}}\left(1 + \frac{1}{2bt} + \frac{1\cdot 3}{(2bt)^2} + \frac{1\cdot 3\cdot 5}{(2bt)^3} + \cdots\right). \quad (552)$$

It is much easier to obtain this series as follows. Expand the operator thus:

$$\left(\frac{1}{b} - \frac{p}{b^2} + \frac{p^2}{b^3} - \cdots\right)p^{\frac{1}{2}}1.$$

Now replace $p^{\frac{1}{2}}1$ by $\frac{1}{\sqrt{\pi t}}1$, and take derivatives term by term exactly as though we dealt with $\frac{1}{\sqrt{\pi t}}$. This yields the above series immediately. This is the procedure of Heaviside. It is much more direct, but unfortunately it sometimes goes violently wrong. Evidently a criterion is needed to show when Heaviside's direct process can be safely employed, and this will be treated presently.

Now consider the operational expression

$$\frac{p}{p+a}\sqrt{p}\,1$$

which can be written as

$$\left(1 - \frac{a}{p+a}\right)\sqrt{p}\,1.$$

The second term has just been expanded asymptotically. So

$$\begin{aligned}\frac{p}{p+a}\sqrt{p}\,1 &= \frac{1}{\sqrt{\pi t}} - \frac{1}{\sqrt{\pi t}}\left(1 + \frac{1}{2at} + \cdots\right)\\ &= -\frac{1}{\sqrt{\pi t}}\left(\frac{1}{2at} + \frac{1\cdot 3}{(2at)^2} + \frac{1\cdot 3\cdot 5}{(2at)^3} + \cdots\right). \quad (553)\end{aligned}$$

This operator is encountered when an exponential e.m.f. ϵ^{-at} is applied to an infinite cable. The full expression is

$$i_0(t) = \sqrt{\frac{c}{r}}\,\frac{p}{p+a}\,\sqrt{p}\;1 \tag{554}$$

so the entering current is obtained by multiplying the series by $\sqrt{\frac{c}{r}}$. Heaviside expanded immediately without splitting into two terms as we have done, and obtained the correct result.

10. **Criteria for an Asymptotic Expansion.** It usually can be determined whether an operational expression admits of an asymptotic expansion by examining the form of the generating function $f(u)$ which specifies the operation. The nature of $f(u)$ as a function of a complex variable determines the properties and limitations of the corresponding operational expression.

In Appendix B are given exact conditions, and if $f(u)$ satisfies these, the asymptotic series is readily obtained and is valid. The conditions, stated with sufficient exactitude for examination of the generating functions usually met with in network analysis, are

(1) $f(u^2)$ is analytic at the origin, so that, it together with all its derivatives is finite and continuous in the immediate vicinity of the origin.

(2) $f(ju)$ is of summable square over the infinite interval, that is

$$\lim_{k\to\infty} \int_{-k}^{k} \left| f(ju) \right|^2 du$$

is finite, and all derivatives of $f(u)$ have this same property. Moreover, it appears from the analysis in Sec. 9 that it is sufficient if $f(u)$ can be expressed as

$$\phi(u) + A + \frac{Bu + C}{u + D}\sqrt{u}$$

where $\phi(u)$ replaces $f(u)$ in criteria (1) and (2), and where A, B, C, and D are any real constants and D is positive.

Thus it is sufficient if the generating function behaves in a certain definite reasonable manner, especially at zero and infinity. If these criteria are met, and if the expansion of $f(u)$ is

$$f(u) = \sum_{1}^{n} a_n u^{\frac{n}{2}}$$

then the asymptotic expansion of $f(p)\ 1$ is given by

$$f(p)\ 1 = f(u) - \frac{a_1}{2\sqrt{\pi t}} + \sum_{1}^{\infty} \frac{a_{2k+1}(-\frac{1}{2})^k[1 \cdot 3 \cdot 5 \cdot \cdots (2k-1)]}{\sqrt{\pi}\, t^{\frac{2k+1}{2}}}. \tag{555}$$

This is exactly the expansion that would be obtained by following the simple rules of Heaviside, but the criteria prevent their use in cases where they do not apply.

Consider the operator which was treated in Sec. 7:

$$f(u) = \frac{u}{\sqrt{u^2 + 2bu}}.$$

It is readily shown that this satisfies the criteria. Hence the asymptotic expansion is exactly as written there.

On the other hand, consider the operational expression:

$$\frac{\omega p}{p^2 + \omega^2} \sqrt{p}\ 1. \tag{556}$$

This does not satisfy Criterion (2) because of the points at which the function becomes infinite; and there is in fact no asymptotic expansion possible for the operational expression as it stands. This is apparent on evaluating by other methods. Yet the scheme of expanding the operator would give a series in descending powers of t. The criteria warn us against the error of employing this scheme.

This operator is encountered in the problem of the application of a sinusoidal e.m.f. to an infinite cable, a constant being here omitted for simplicity. A solution in closed form can be written without difficulty by the use of Borel's theorem. Without a formal solution, however, it is apparent physically that the current entering the cable ultimately subsides to a steady sinusoid, so the reason a valid

asymptotic solution cannot be written is immediately evident.

The reason this operator does not satisfy our criteria is the presence of singularities at $\pm j\omega$. Instead of this operator, therefore, consider for the moment,

$$\left[\frac{\omega p^{\frac{1}{2}}}{p^2+\omega^2}-\frac{\frac{\omega}{2}\sqrt{-j\omega}}{p+j\omega}-\frac{\frac{\omega}{2}\sqrt{j\omega}}{p-j\omega}\right]1.$$

It will be noted that two terms have been subtracted, each constructed in such a way as to just remove one of the singularities. This will be compensated for in the final result by again adding the result of the operator here subtracted. Combining the terms now being examined, there is obtained

$$\frac{\omega p^{\frac{1}{2}}-\frac{\omega^{\frac{3}{2}}}{\sqrt{2}}(p-\omega)}{p^2+\omega^2}1 \tag{557}$$

which satisfies the criteria. The asymptotic series is now readily obtained by expanding in power series, but considerable labor can be saved by noting that the first term of the numerator contributes all of the terms which yield terms in the asymptotic expansion, except for the constant term obtained by inserting $p=0$. The asymptotic expansion of (557) is thus obtained from

$$\sqrt{\frac{\omega}{2}}+\left(\frac{p}{\omega}-\frac{p^3}{\omega^3}+\frac{p^5}{\omega^5}-\cdots\right)p^{\frac{1}{2}}1$$

which yields

$$\sqrt{\frac{\omega}{2}}-\sqrt{\frac{\omega}{\pi}}\left[\frac{1}{2\,(\omega t)^{\frac{3}{2}}}-\frac{1\cdot3\cdot5}{2^3\,(\omega t)^{\frac{7}{2}}}+\frac{1\cdot3\cdot5\cdot7\cdot9}{2^5\,(\omega t)^{\frac{11}{2}}}-\cdots\right].$$

To this must be added the result of

$$\frac{\omega^{\frac{3}{2}}}{\sqrt{2}}\,\frac{p-\omega}{p^2+\omega^2}1$$

which is

$$\sqrt{\frac{\omega}{2}}(\sin \omega t + \cos \omega t - 1).$$

The final result is then

$$\frac{\omega p}{p^2 + \omega^2}\sqrt{p}\,1 = \sqrt{\frac{\omega}{2}}(\sin \omega t + \cos \omega t) - \frac{1}{\sqrt{\pi t}}\left(\frac{1}{2\,\omega t} - \frac{1 \cdot 3 \cdot 5}{(2\,\omega t)^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{(2\,\omega t)^5} - \cdots\right).$$

In this example we have illustrated an operational procedure which is of considerable utility. In general if a generating function has singularities which prevent an asymptotic expansion, these can oftentimes be usefully removed by subtracting a simple function designed to remove the difficulty. Where the actual result is a sinusoid plus a function admitting of asymptotic treatment, this process yields both the steady-state term and the asymptotic series. It is similarly possible to attack cases where the sinusoid appears as one factor in a product.

11. **Infinite Cable with Terminal Resistance.** Let this treatment be applied to the problem of an infinite cable with a series resistance connected at the home end, Fig. 49, and to which a unit e.m.f. is suddenly applied. The impedance function of the cable is $\sqrt{\frac{r}{cp}}$, and hence that of the combination is

$$R + \sqrt{\frac{r}{cp}}.$$

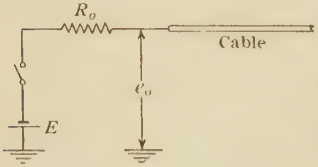

FIG. 49. Infinite Cable with Terminal Resistance.

So the entering current becomes

$$i_0(t) = \frac{1}{R + \sqrt{\frac{r}{cp}}}\,1 = \sqrt{\frac{c}{r}}\,\frac{p^{\frac{1}{2}}}{1 + h^{\frac{1}{2}}\,p^{\frac{1}{2}}}\,1 \qquad (558)$$

where

$$h^{\frac{1}{2}} = R\sqrt{\frac{c}{r}}.$$

Consider first the convergent expansion. The expansion of the operator in descending powers of p gives

$$\begin{aligned} i_0(t) &= \frac{1}{R}\left(1 - \frac{1}{(hp)^{\frac{1}{2}}} + \frac{1}{(hp)} - \frac{1}{(hp)^{\frac{3}{2}}} + \cdots\right)1 \\ &= \frac{1}{R}\left(1 + \frac{1}{hp} + \frac{1}{(hp)^2} + \cdots\right)1 \\ &\quad - \frac{h^{\frac{1}{2}}}{R}\left(\frac{1}{hp} + \frac{1}{(hp)^2} + \cdots\right)p^{\frac{1}{2}}\,1. \end{aligned} \tag{559}$$

Now replace $p^{\frac{1}{2}}\,1$ by $\dfrac{1}{\sqrt{\pi t}}$ and integrate:

$$\begin{aligned} i_0(t) &= \frac{1}{R}\left(1 + \frac{t}{h} + \frac{t^2}{2!\,h^2} + \cdots\right) \\ &\quad - \frac{2}{R\sqrt{\pi}}\left[\left(\frac{t}{h}\right)^{\frac{1}{2}} + \frac{2}{3}\left(\frac{t}{h}\right)^{\frac{3}{2}} + \frac{2^2}{3\cdot 5}\left(\frac{t}{h}\right)^{\frac{5}{2}} + \cdots\right]. \end{aligned} \tag{560}$$

This may be written

$$i_0(t) = \frac{1}{R}\,\epsilon^{\frac{t}{h}} - \frac{2}{R}\sqrt{\frac{t}{\pi h}}\left[1 + \frac{2}{3}\left(\frac{t}{h}\right) + \frac{2^2}{3\cdot 5}\left(\frac{t}{h}\right)^2 + \cdots\right]. \tag{561}$$

This convergent result, following Volterra, is convenient for computation when t is small. Although it is valid for any value of t, it is enormously hard to compute from when t is at all large. Although the first term is a positive exponential, the current does not increase exponentially, for the two terms nearly cancel when t is large and the current actually tapers off to zero just as it does when the resistance is left off. This will be seen from the asymptotic expansion. The reason that a positive exponential term may be encountered in this treatment will appear more clearly in the next chapter.

To obtain the asymptotic expansion, note that the generating function satisfies the Wiener conditions. So the operator can be expanded in ascending powers of p:

$$i_0(t) = \frac{1}{R}[(hp)^{\frac{1}{2}} - (hp) + (hp)^{\frac{3}{2}} - (hp)^2 + \cdots]\, 1. \quad (562)$$

The integral powers of p may now be dropped, leaving

$$i_0(t) = \frac{h^{\frac{1}{2}}}{R}[1 + (hp) + (hp)^2 + \cdots]p^{\frac{1}{2}}\, 1 \quad (563)$$

so

$$i_0(t) = \sqrt{\frac{c}{\pi t}}\left(1 - \frac{h}{2t} + \frac{3h^2}{2^2 t^2} - \frac{3 \cdot 5 h^3}{2^3 t^3} + \cdots\right). \quad (564)$$

This is the asymptotic solution, excellent for computation when t is large. It is immediately apparent that the current finally tapers off just as though the resistance were not present.

We have here followed the Heaviside procedure in obtaining the asymptotic expansion, especially in dropping the integral powers of p. The justification for so doing is contained in expression (555). Notice that in applying this the integral powers are omitted, and only those terms retained which have been included above. Heaviside often dropped series in integral powers of p applied to the unit function. This is not justifiable in general; in fact, one can make serious errors by so doing. When, however, a generating function satisfies our criteria for an asymptotic expansion, the procedure of dropping them is correct. This is of course connected with the statement that the asymptotic expansion of a rational algebraic operational expression, if it exists at all, is simply a string of zeros.

12. **Treatment by Successive Approximations. Cable with Terminal Inductance.** The physical reasoning behind the series expansion of operators will become more apparent if we proceed to obtain such a development by a

scheme of successive approximations based on detailed physical reasoning. Such a device is often used in the solution of physical problems,[1] and it often makes it possible to arrive at an objective when direct frontal attack is out of the question. It consists in general in obtaining a solution under simplifying assumptions, solving for the effect of the ignored factors, making a new simplified solution on this basis, and so on. Operational processes are useful in rapidly arriving at the approximations in solving circuit problems in this manner. The process will be met again in connection with networks with variable parameters. The method can best be explained by the use of an example. For this purpose we shall choose a problem which gave Heaviside much difficulty, for reasons which will be considered. This is the problem of a cable with terminal inductance. The resistance of the inductance coil will be neglected, although its effect can be included readily. A unit e.m.f. is suddenly applied to the cable and inductance in series (Fig. 50) and the voltage at the home end of the cable is to be obtained. The impedance function of coil and cable in series is

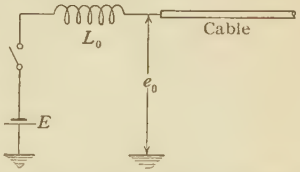

FIG. 50. Infinite Cable with Terminal Inductance.

$$Z(p) = Lp + \sqrt{\frac{r}{cp}}. \tag{565}$$

So the entering current is

$$i_0(t) = \frac{1}{Lp + \sqrt{\frac{r}{cp}}}\, 1. \tag{566}$$

[1] For an excellent example of the method see H. B. Dwight, Skin Effect and Proximity Effect in Tubular Conductors, Trans. A.I.E.E., **41**, 1922, p. 189; Proximity Effect in Thin Tubes, Trans. A.I.E.E., **42**, 1923, p. 850.

Consider the voltage at the end of the cable, which is the applied voltage minus the drop in the coil; that is,

$$e_0(t) = 1 - \frac{Lp}{Lp + \sqrt{\frac{r}{cp}}}\,1$$

$$= \frac{1}{1 + Lp\sqrt{\frac{cp}{r}}}\,1. \tag{567}$$

This is an operator of the form:

$$\frac{1}{1 + (bp)^{\frac{1}{2}}} \tag{568}$$

where

$$b = \left(L\sqrt{\frac{c}{r}}\right)^{\frac{2}{3}}.$$

Here we note a peculiar matter. If the expression be written:

$$e_0(t) = \frac{1}{Z(p)}\,1 = \frac{1}{1 + (bp)^{\frac{1}{2}}}\,1 \tag{569}$$

then

$$Z(p) = 1 + (bp)^{\frac{1}{2}} \tag{570}$$

and this impedance function has three roots:

$$\frac{1}{b},\ \frac{1}{b}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right),\ \frac{1}{b}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

that is $1/b$ times the three cube roots of unity. One of these roots is a positive real. Yet a dissipative network is being dealt with. The difficulty is not avoided if the resistance of the coil is included. In fact it appeared in the previous problem of a simple resistance in series with the cable, for the operational formula for entering current then was

$$i_0(t) = \frac{1}{R + \sqrt{\frac{r}{cp}}}\,1 \tag{571}$$

and the only root was $\frac{r}{cR^2}$, which was likewise positive. Of course these positive roots appear only because there is an ambiguity of sign of the radical, and they are physically spurious. The systems have no natural frequency corresponding to them, and there can be no term in a correct solution which increases without limit. Mathematically, however, this does not help a bit, for mathematically the positive roots are there and our treatment has generally been on the assumption that there are no positive roots. So there is no demonstration of the expansion theorem or of any of our usual operational formulas for a case like this. A further discussion of the meaning and treatment of positive roots is given in the next chapter, but this example is a good illustration of a way of avoiding difficulty.

We know that the network with which we deal is actually linear and dissipative, and superposition can be used. A solution, good for computation for small t, can be found by successive approximations. Examine the problem with the coil in series. In the first instant the entire applied e.m.f. is brought to bear on the coil alone, producing the current

$$i(t) = \frac{1}{L}\, t\, 1.$$

The voltage at the home end of the cable is obtained by applying to this the impedance operator for the cable:

$$\begin{aligned}
\frac{1}{L}\sqrt{\frac{r}{cp}}\, t\, 1 &= \frac{1}{Lp}\sqrt{\frac{r}{cp}}\, 1 \\
&= \frac{1}{L}\sqrt{\frac{r}{c}}\, p^{-\frac{1}{2}}\, 1 \\
&= \frac{4}{3L}\sqrt{\frac{r}{c\pi}}\, t^{\frac{3}{2}}\, 1 \\
&= \frac{4}{3\sqrt{\pi}\, b^{\frac{1}{2}}}\, t^{\frac{3}{2}}\, 1 \qquad (572)
\end{aligned}$$

A second approximation can now be obtained by writing the voltage on the coil as

$$1 - \frac{1}{Lp}\sqrt{\frac{r}{cp}}\,1$$

and hence the current as

$$\frac{1}{Lp}\left(1 - \frac{1}{Lp}\sqrt{\frac{r}{cp}}\,1\right)$$

and the end voltage of the cable as

$$\frac{1}{Lp}\sqrt{\frac{r}{cp}}\left(1 - \frac{1}{Lp}\sqrt{\frac{r}{cp}}\,1\right)$$
$$= \frac{4}{3\sqrt{\pi}\,b^{\frac{3}{2}}}t^{\frac{3}{2}} - \frac{1}{3!\,b^3}t^3. \tag{573}$$

Proceeding in this manner,

$$e_0(t) = \frac{2^2}{\sqrt{\pi}\,3}\left(\frac{t}{b}\right)^{\frac{3}{2}} - \frac{1}{3!}\left(\frac{t}{b}\right)^3 + \frac{2^5}{\sqrt{\pi}\,3\cdot 5\cdot 7\cdot 9}\left(\frac{t}{b}\right)^{\frac{5}{2}} - \frac{1}{6!}\left(\frac{t}{b}\right)^6 + \cdots. \tag{574}$$

Notice that this process is really a scheme for getting the various derivatives of the result at $t = 0$. It is an absolutely convergent result, and could reasonably be expected to be valid for any value of t, although it is useless for computation with t large. It is in fact exactly the same result[1] that is obtained by expanding the operator in descending powers of p and integrating term by term. It is evident that the method of successive approximations does not constitute a time-saving process here, although it may render clear the meaning of the successive terms in the result.

An asymptotic result can be obtained also by successive approximations, and the procedure will serve the same pur-

[1] Heaviside, Electromagnetic Theory, Vol. II, p. 42.

pose. When a long time has elapsed the voltage at the home end of the cable will evidently be nearly unity. But if this voltage had been present from $t = 0$, the current would have been

$$i_0 = \sqrt{\frac{c}{r}} p^{\frac{1}{2}} 1 = \sqrt{\frac{c}{\pi r}} t^{-\frac{1}{2}}. \tag{575}$$

Actually, of course, the voltage at the home end of the cable rises more or less rapidly after the switch is closed, and approaches unity. It may be considered as made up of two parts, one of which has the constant value unity at all times after the switch is closed, and the other one of which is minus unity at $t = 0$ and dies away soon thereafter. The current term corresponds to the first part of this voltage. Any current caused by the second part will have died away to zero after a sufficient lapse of time. So the above is an approximate expression for the way the current tapers off. However, yet a better one can be obtained. This first approximation was found by neglecting the effect of the coil and considering the direct effect of the source only. Actually, if the current in the cable varied in accordance with the above, there would be a drop in the coil:

$$Lp\sqrt{\frac{c}{\pi r}} t^{-\frac{1}{2}} 1 = L\sqrt{\frac{c}{r}} p^{\frac{1}{2}} 1 \tag{576}$$

but in order to have caused such a current to flow through coil and cable, the source would have had to furnish a voltage:

$$1 + L\sqrt{\frac{c}{r}} p^{\frac{1}{2}} 1 = 1 - \frac{b^{\frac{1}{2}}}{2\sqrt{\pi}} t^{-\frac{1}{2}}. \tag{577}$$

In forming these expressions note that, since the current given by the first assumption (575) is always decreasing when $t > 0$, the drop in the coil (576) is in the direction which adds to the source, so that the sign of the last term of (577)

is minus. If the current caused by a source of amount:

$$-L\sqrt{\frac{c}{r}}p^{\frac{3}{2}}\,1 = +\frac{b^{\frac{3}{2}}}{2\sqrt{\pi}}t^{-\frac{3}{2}} \tag{578}$$

could be found and added to the first term of current, a complete correct solution would result. This cannot be done directly, but it can be approximated by again neglecting the coil and examining the current caused by such a source acting on the cable alone. It will be most convenient to keep the expressions in operational form. The second approximation for current is thus

$$\sqrt{\frac{c}{r}}p^{\frac{1}{2}}\left(-L\sqrt{\frac{c}{r}}p^{\frac{3}{2}}\,1\right) = -L\frac{c}{r}p^2\,1. \tag{579}$$

This process is continued. Such a current term would require at the source a voltage term:

$$Lp\left(-L\frac{c}{r}p^2\,1\right) = -L^2\frac{c}{r}p^3\,1 \tag{580}$$

and the negative of this applied to the cable alone gives a current term:

$$\sqrt{\frac{c}{r}}p^{\frac{1}{2}}\left(L^2\frac{c}{r}p^3\,1\right) = L^2\left(\frac{c}{r}\right)^{\frac{3}{2}}p^{\frac{7}{2}}\,1 \tag{581}$$

and so on. The complete current expression is then

$$i_0 = \sqrt{\frac{c}{r}}p^{\frac{1}{2}}\,1 - L\frac{c}{r}p^2\,1 + L^2\left(\frac{c}{r}\right)^{\frac{3}{2}}p^{\frac{7}{2}}\,1 - \cdots. \tag{582}$$

From this the voltage at the home end of the cable is readily obtained.

$$\begin{aligned} e_0 &= 1 - Lpi_0 \\ &= 1 - L\left(\frac{c}{r}\right)^{\frac{1}{2}}p^{\frac{3}{2}}\,1 + L^2\frac{c}{r}p^3\,1 - L^3\left(\frac{c}{r}\right)^{\frac{3}{2}}p^{\frac{9}{2}}\,1 + \cdots. \end{aligned} \tag{583}$$

Evaluating the portion of this which involves fractional-order derivatives, we have

$$e_0 = 1 + \frac{1}{2\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{1}{2}} - \frac{3\cdot 5\cdot 7}{2^4\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{3}{2}} + \frac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{2^7\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{13}{2}} - \cdots \tag{584}$$

which is a solution good for large values of t.

The operator of our problem satisfies the Wiener conditions, and on expanding we obtain exactly, and of course much more rapidly, the solution written here. So (584) is actually an asymptotic solution for the problem, and the treatment by successive approximations is useful only as it serves to render the ideas involved more clear. However there is still much of interest in this problem which has not yet been treated.

Heaviside first expanded the operator in series directly and, on applying it term by term, obtained the convergent expansion (574) directly. Heaviside also obtained a second solution — the asymptotic series (584) plus a term

$$-\frac{4}{3}\epsilon^{-\frac{t}{2b}}\cos\frac{\sqrt{3}}{2b}t$$

although apparently he does not state, there or elsewhere, how he did it. He writes that its derivation involved advanced operational ideas, which is certainly true. There is, as has been shown, little to be gained by adding an exponential term to an asymptotic series, except where it comes from a separate factor in the problem, and where its value for some values of t may be greater than the error of the asymptotic series itself. It will be worth while to examine whether such is the case in the problem we have in hand.

Rewrite the operator (568) as

$$\frac{2}{1-(bp)^3} - \frac{1}{1-(bp)^{\frac{3}{2}}}. \tag{585}$$

It will be noted by performing the indicated division that the first fraction corresponds to a constant term plus a part that is quite similar to the portion that was discarded in obtaining the asymptotic solution from (583). The second fraction, taken together with this constant term, will yield the asymptotic solution. If an asymptotic development is written for the first fraction it will consist of the constant term alone. But it is not necessary to interpret both fractions in the same way, that is by asymptotic developments, and the result can be improved by proceeding otherwise.

As will be treated more in detail in the next chapter, it is essential, when dealing with an operator of this sort, which involves spurious positive roots, to use consistent definitions in evaluating each part. It will not do at all to interpret one part by Wiener's treatment to obtain an asymptotic series, and the other part by some operational scheme which is inconsistent therewith. The first term of (585) must therefore be handled by methods which are in agreement with the process by which the second term has been developed into an asymptotic series. Such a method, as will appear in the next chapter, is the expansion theorem, used in proper manner in view of the presence of the positive root. This, in brief, consists of using the theorem as previously obtained, but leaving out any term corresponding to a root with a positive real portion.

The impedance function of the first fraction of (585) is

$$Z(p) = \frac{1}{2}[1 - (bp)^3]$$

and the roots of $Z(p) = 0$ are hence

$$\frac{1}{b},\ \frac{1}{b}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right),\ \frac{1}{b}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right).$$

There is also readily obtained

$$Z(0) = \frac{1}{2}$$

$$Z'(p) = -\frac{3}{2}b^3p^2.$$

The expansion theorem, ignoring the root $1/b$, thus yields, on substituting:

$$2 - \frac{2}{3}\epsilon^{-\frac{t}{2b}}\epsilon^{\frac{j\sqrt{3}}{2b}t} - \frac{2}{3}\epsilon^{-\frac{t}{2b}}\epsilon^{-\frac{j\sqrt{3}}{2b}t}$$

or

$$2 - \frac{4}{3}\epsilon^{-\frac{t}{2b}}\cos\frac{\sqrt{3}}{2b}t \tag{586}$$

and this, together with the asymptotic series obtained from the second fraction of (585) is exactly what Heaviside wrote.[1]

The operator in the form (585) thus yields the asymptotic series (584) together with a damped sinusoid. Thus by one process of anlaysis the damped sinusoid is omitted, and by an equally legitimate process it is included in the result. There is no inconsistency here, because the asymptotic expansion of the damped sinusoid is zero, and either expression is quite correct provided t is taken large enough, which is all that can be expected of an asymptotic result. It is an entirely different question whether the extra term should be included in order to give an accurate solution of the problem for values of t which are not very large. Unfortunately there is no criterion at present available by which this question can be settled mathematically. It is probably obtainable by an examination of the extended summability of the series, but such an analysis has not yet been carried through. A workable heuristic rule appears to be to split off as much of a term as possible before making the asymptotic development, and then interpret as has been done above.

These two examples, of the cable in series with resistance and inductance respectively, illustrate in excellent manner the power and the limitations of the asymptotic expansion. In Fig. 51 and Fig. 52 are plotted the results, those obtained both from the convergent expansion and from the asymptotic expansion. These curves were plotted from the following tabulated computations:

[1] Electromagnetic Theory, Vol. II, p. 44.

CABLE WITH TERMINAL RESISTANCE

t/b	e_0 Curve I	No. of terms	e_0 Curve II	No. of terms	e_0 Curve III	No. of terms
0.1	0.2764	5				
1.0	0.5724	10	0.710	3	0.128	4
3.0	0.7127	17	0.7240	4	0.6976	5
5.0	0.7676	23	0.7689	6	0.7662	7
10.0			0.82942	11	0.82943	12

I. Computed by convergent series:

$$e_0 = -\left(e^{\frac{t}{b}} - 1\right) + \frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{b}}\left[1 + \frac{2}{3}\left(\frac{t}{b}\right) + \frac{2^2}{3\cdot 5}\left(\frac{t}{b}\right)^2 + \cdots\right].$$

II. Computed by asymptotic series:

$$e_0 = 1 - \frac{1}{\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{1}{2}}\left[1 - \frac{1}{2}\left(\frac{b}{t}\right) + \frac{3}{2^2}\left(\frac{b}{t}\right)^2 - \cdots\right]$$

stopping at the smallest term.

III. Same as II except one more term added to the series.

CABLE WITH TERMINAL INDUCTANCE

t/b	e_0 Curve I	e_0 Curve I′	e_0 Curve II		e_0 Curve III	
0.5	0.25					
1.0	0.603	0.6034				
2.0	1.152					
3.0	1.297	1.2999	1.029	1.191	1.285	1.447
4.0	1.208					
5.0	1.07	1.0644	1.0226	1.0260	1.0634	1.0668
6.0	0.97					
8.0			1.0121	1.0126	0.9926	0.9931

I. Computed by Heaviside using convergent series.

I′. Points checked by the writer using convergent series.

II. Asymptotic series (584). The lower values are obtained by stopping at the smallest term, while the upper curve is the result of adding one more term in the series.

III. Asymptotic series with exponential term:

$$e_0 = 1 - \frac{4}{3}\epsilon^{-\frac{1}{2b}t}\cos\frac{\sqrt{3}}{2b}t + \frac{1}{2\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{3}{2}}\left[1 - 1\cdot 3\cdot 5\cdot 7\left(\frac{b}{2t}\right)^3 + \cdots\right].$$

Lower curve obtained by stopping at the smallest term, as in II.

In Figs. 51 and 52 the magnitude of the uncertainty in using the asymptotic series is shown by the shaded area. It is very easy to compute results for t small from the convergent series, or for t large from the asymptotic series. When t is at all large, the labor involved in calculating from the convergent series is indeed great. When t is small, the uncertainty of the asymptotic series prevents precision.

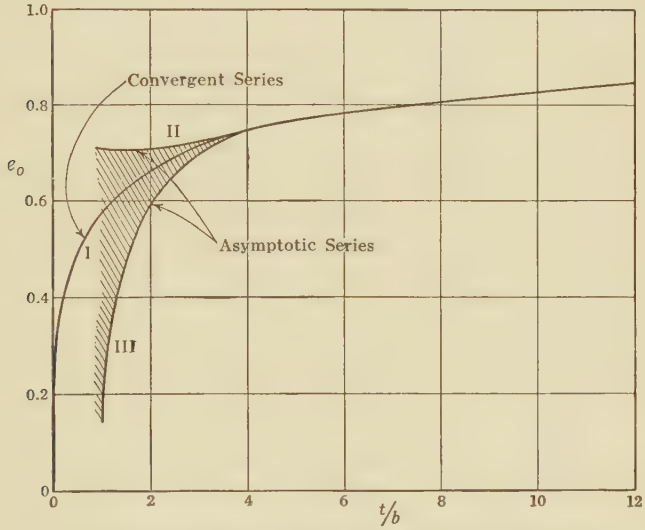

FIG. 51. Terminal Voltage of Cable with Terminal Resistance, Computed from Convergent and Asymptotic Series.

Moreover there is a very interesting matter which appears. In the case of the series resistance, the two curves merge excellently. It is apparent that either series represents the true result for the problem in hand for any value of t, although of course the uncertainty involved in using the asymptotic series does not become negligible until t is rather large. In the case of the series inductance, on the other hand, the two curves do not merge. That obtained from the convergent expansion oscillates while that for the asymptotic series does

not. The latter is certainly the asymptotic series which corresponds to the operational expression of the problem, and it hence gives the result of the problem within any precision desired, provided t is taken large enough. But

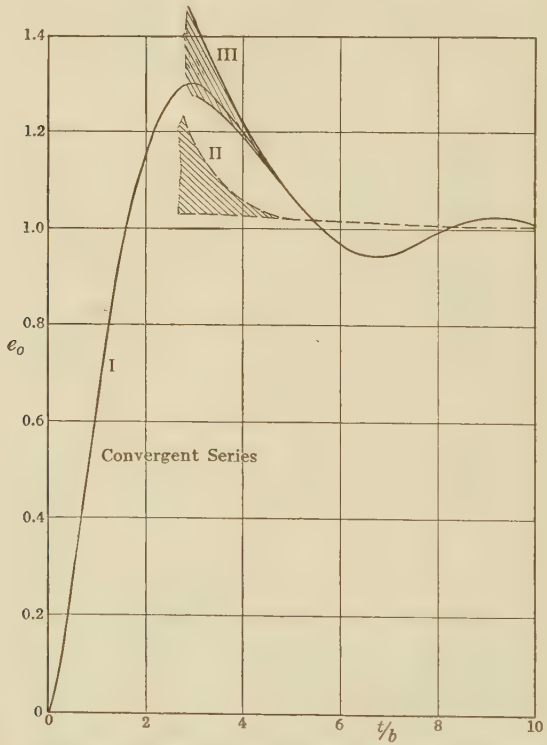

FIG. 52. Terminal Voltage of Cable with Terminal Inductance, Computed from Convergent and Asymptotic Series, and by Including a Damped Sinusoidal Term.

there is a region — a range of t — within which the asymptotic series does not give a correct result to the problem, even to the precision of its degree of uncertainty, and the result from the convergent series oscillates about the curve given by the asymptotic series. But if we add to the

asymptotic series the value of the damped sinusoid written above, then the result checks numerically in the intermediate region with the result of the convergent expansion. Evidently the extra term was therefore needed, not to afford a proper asymptotic series, but to give a correct result within the range of uncertainty of the asymptotic series when it is used for computation for moderate values of t. It should be emphasized, however, that there is no criterion available for determining whether the process here used yields the correct additional term both in form and magnitude. This is a gap which still exists in operational treatment.

13. **The Fractional-power Exponential Operator. Series in Integral Powers of p.** There are rare cases when series in integral powers of p applied to the unit function are met in the course of obtaining convergent solutions. When this occurs difficulty can be avoided by treating the operator in some other manner than by series expansion. However, this sort of operational problem will be studied briefly, for there are some ideas involved that are decidedly interesting.

Consider as an example about the only one that is met with in treating ordinary networks which illustrates the matter in hand — the operational expression

$$\epsilon^{-\beta x}\, 1 \tag{587}$$

where $\beta = \sqrt{rcp}$. This is the operator which gives the voltage at a point x on a cable when unit e.m.f. is applied at the home end. Abbreviate by

$$\gamma = rcx^2$$

and this is

$$\epsilon^{-\sqrt{\gamma p}}\, 1.$$

Expand the exponential:

$$\left(1 - \gamma^{\frac{1}{2}}p^{\frac{1}{2}} + \frac{\gamma p}{2!} - \frac{\gamma^{\frac{3}{2}}p^{\frac{3}{2}}}{3!} + \cdots\right) 1$$

which may be written

$$\left(1+\frac{\gamma p}{2!}+\frac{\gamma^2 p^2}{4!}+\cdots\right)1$$
$$-\gamma^{\frac{1}{2}}\left(1+\frac{\gamma p}{3!}+\frac{\gamma^2 p^2}{5!}+\cdots\right)p^{\frac{1}{2}}\,1.$$

Now if the first series is discarded except for the first term, and the second series treated just as before, the result is

$$\epsilon^{-\sqrt{\gamma p}}\,1 = 1-\sqrt{\frac{\gamma}{\pi}}\left(t^{-\frac{1}{2}}-\frac{1}{2}\frac{\gamma}{3!}t^{-\frac{3}{2}}+\frac{1}{2}\frac{3}{2}\frac{\gamma^2}{5!}t^{-\frac{5}{2}}-\cdots\right) \quad (588)$$

or for the voltage at a point x on the cable:

$$\epsilon^{-\sqrt{rcx^2p}}\,1 = 1-x\sqrt{\frac{rc}{\pi t}}\left[1-\frac{1}{3}\left(\frac{rcx^2}{4t}\right)+\frac{1}{5\cdot 2!}\left(\frac{rcx^2}{4t}\right)^2-\cdots\right]$$
$$=1-\frac{2}{\sqrt{\pi}}\left[\left(\frac{rcx^2}{4t}\right)^{\frac{1}{2}}-\frac{1}{3}\left(\frac{rcx^2}{4t}\right)^{\frac{3}{2}}+\frac{1}{5\cdot 2!}\left(\frac{rcx^2}{4t}\right)^{\frac{5}{2}}-\cdots\right]. \quad (589)$$

The result is a convergent series and it is particularly convenient for use when t is large. This series was recognized by Heaviside as that of the error function. This function, which occurs often in the theory of probability, is $2/\sqrt{\pi}$ times the area under the probability curve:

$$\int_0^y \epsilon^{-y^2}\,dy = y-\frac{y^3}{3}+\frac{y^5}{5\cdot 2!}-\frac{y^7}{7\cdot 3!}+\cdots. \quad (590)$$

If

$$y=\sqrt{\frac{\gamma}{4t}}=\sqrt{\frac{rcx^2}{4t}}$$

$$\epsilon^{-\sqrt{\gamma p}}\,1 = 1-\frac{2}{\sqrt{\pi}}\int_0^y \epsilon^{-y^2}\,dy \quad (591)$$

which can be expressed as

$$\epsilon^{-\sqrt{\gamma p}}\,1 = 1-erf\sqrt{\frac{\gamma}{4t}} \quad (592)$$

where *erf* means "error function."[1] Since there are tables of this integral,[2] it is excellent for computation. For the cable, therefore, when unit electromotive force is applied,

$$e = \left(1 - erf\frac{x}{2}\sqrt{\frac{rc}{t}}\right). \tag{593}$$

Now this particular operational expression was evaluated in Chapter XI by entirely rigorous processes. We obtained

$$\epsilon^{-\sqrt{\gamma p}}\,1 = \tfrac{1}{2}\sqrt{\frac{\gamma}{\pi}}\int_0^t \lambda^{-\frac{3}{2}}\,\epsilon^{-\frac{\gamma}{4\lambda}}\,d\lambda. \tag{594}$$

Make the substitution[3] $\lambda = \gamma\frac{\zeta}{4}$, and the integral becomes

$$\frac{1}{\sqrt{\pi}}\int_0^{\frac{4t}{\gamma}} \zeta^{-\frac{3}{2}}\,\epsilon^{-\frac{1}{\zeta}}\,d\zeta$$

which may be written

$$\frac{1}{\sqrt{\pi}}\int_0^{\infty} \zeta^{-\frac{3}{2}}\,\epsilon^{-\frac{1}{\zeta}}\,d\zeta - \frac{1}{\sqrt{\pi}}\int_{\frac{4t}{\gamma}}^{\infty} \zeta^{-\frac{3}{2}}\,\epsilon^{-\frac{1}{\zeta}}\,d\zeta.$$

The value of the first integral is known[4] to be $\sqrt{\pi}$. Expand the exponential in the second in the usual power series:

$$1 - \frac{1}{\sqrt{\pi}}\int_{\frac{4t}{\gamma}}^{\infty}\left(\frac{1}{\zeta^{\frac{3}{2}}} - \frac{1}{\zeta^{\frac{5}{2}}} + \frac{1}{2!\,\zeta^{\frac{7}{2}}} - \cdots\right)d\zeta.$$

Integration term by term gives series (589). This justifies the process by which it was obtained there by immediate application of the operator in a series. In Chapter XIV

[1] Sometimes *erf* is used as an abbreviation for the integral alone, without the constant, but the notation above appears preferable. See Jeffreys, Operational Methods, Cambridge University Tract, p. 94.

[2] B. O. Peirce, Short Table of Integrals, pp. 116–120.
Heaviside, Electromagnetic Theory, Vol. II, p. 51.

[3] Carson, *loc. cit.*

[4] Bierens de Haan, p. 122.

another series will be given for this error-function result which is sometimes more convenient for computation.

It will be of interest to study somewhat further this matter of dropping a series of integral powers of p acting on the unit function,[1] although the treatment is not needed for justification of the result just obtained, or in connection with asymptotic developments. Again a somewhat clearer physical picture of what is being done may be gained by discussing, without any pretence at rigor, just why it is that such a series gives at times a zero result. Heaviside often discarded such a series, placing it equal to zero for positive values of t, with a remark to the effect that $p^k\ 1$, where k is a positive integer, is zero, for it then represents the derivative of a constant. This is undoubtedly all right when there is only a single term of this sort in a result, for it then represents only an impulse at $t = 0$; but it is an entirely different question when there is an infinite series of such terms. As Heaviside remarked, "there are zeros and zeros," and he sometimes replaced such a series by a finite expression in obtaining a result. The continuous function,

$$f(t) = \tfrac{1}{2}\,(1 + \tanh kt)$$

is a good approximation for the unit function if k is large (Fig. 53). Let us find the result of applying $\Sigma a_n p^n$ to this, and then let k increase without limit. We should arrive at $\Sigma a_n p^n\ 1$. The various derivatives are

$$\begin{aligned}
pf\,(t) &= \tfrac{1}{2}\,k\,\operatorname{sech}^2 kt \\
p^2 f\,(t) &= -k^2 \operatorname{sech}^2 kt \tanh kt \\
p^3 f\,(t) &= 2\,k^3 \operatorname{sech}^2 kt \tanh^2 kt - k^3 \operatorname{sech}^4 kt \\
p^4 f\,(t) &= -2^2 k^4 \operatorname{sech}^2 kt \tanh^3 kt + 2^3 k^4 \operatorname{sech}^4 kt \tanh kt \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

[1] Stranco, following the methods of Giorgi, has developed a criterion for use in this connection.

L. Straneo, Sulle condizioni di validità di alcuni sviluppi degli operatori funzionali, Atti della Reale Accademia Nazionale dei Lincei, Rome, **6**, 1927, p. 292.

Now for large k, tanh kt approaches one, and sech kt approaches $2\,\epsilon^{-kt}$. So

$$\Sigma a_n p^n\, 1 = \lim_{k\to\infty} \epsilon^{-2kt}[a_1(2k) - a_2(2k)^2 + a_3(2k)^3 - \cdots]$$
$$-\lim_{k\to\infty} \tfrac{1}{4}\,\epsilon^{-4kt}[a_3(4k)^3 - 2\,a_4(4k)^4 + \cdots]$$
$$\cdots\cdots\cdots\cdots\cdots \qquad (595)$$

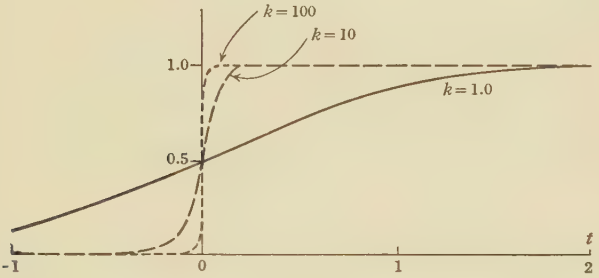

FIG. 53. Approximation for the Unit Function, $\frac{1}{2}(1+\tanh kt)$.

It appears reasonable that any series of coefficients which would make the first line of this expression zero, would also make the other terms vanish. Apparently then we may have integral power series which, applied to the unit function, will yield a result which it would appear reasonable to put equal to zero. This can be examined by considering

$$\lim_{k\to\infty} \epsilon^{-kt}\,(a_1k - a_2k^2 + a_3k^3 - \cdots) \qquad (596)$$

although, as has been noted, the real justification is the treatment reviewed above. However, let us look at an example or two. For the simple inductive circuit,

$$\frac{1}{p+b}\,1 = \left\{\frac{1}{b} - \frac{1}{b}\left[\frac{p}{b} - \frac{p^2}{b^2} + \frac{p^3}{b^3} - \cdots\right]\right\} 1. \qquad (597)$$

If terms involving p are omitted, we shall have the absurd result $\frac{1}{b}$. The test series (596) becomes, when the coeffi-

cients of the series in the bracket of (597) are inserted,

$$\lim_{k\to\infty} \epsilon^{-kt}\left(\frac{k}{b}+\frac{k^2}{b^2}+\frac{k^3}{b^3}+\cdots\right).$$

We should certainly not be tempted to set this equal to zero, even if we did not have the guide of a known result to warn us.

Consider also,

$$\epsilon^{-bp}\,1.$$

Write this

$$\left(1-bp+\frac{b^2p^2}{2!}-\cdots\right)1 = 1-\left(bp-\frac{b^2p^2}{2!}+\cdots\right)1 \tag{598}$$

and the value of the test series for the series in the parenthesis is

$$\lim_{k\to\infty} \epsilon^{-kt}\left(bk+\frac{b^2k^2}{2!}+\cdots\right).$$

A constant can be added in the parenthesis without affecting the limit; so we examine

$$\lim_{k\to\infty} \epsilon^{-kt}\left(1+bk+\frac{b^2k^2}{2!}+\cdots\right).$$

But this is

$$\lim_{k\to\infty} \epsilon^{-kt}\,\epsilon^{+bk} = \lim_{k\to\infty} \epsilon^{-k(t-b)}.$$

As far as this single series is concerned, there is thus no indication that we shall go wrong if $\sum a_n p^n\,1$ is here set equal to zero for t greater than b. That is, in expression (598) all but the first term may be excluded for $t > b$. Thus

$$\epsilon^{-bp}\,1 = \text{unity}, \quad t > b.$$

But it is known that

$$\epsilon^{-bp}\,1 = 1_b$$

so this is a correct result. Similarly, ϵ^{bp} can be set equal to unity when $t > -b$. This is an indication that it was permissible to write $\epsilon^{bp}\, 1 = 1_{-b}$.

The discussion here, it should be emphasized, is for purposes of illustration, and not for justification of steps which have been established elsewhere. From this same point of view another step which occurred in series developments may be examined. This is the process of applying a series of integral powers of p to $t^{-\frac{1}{2}}\, 1$ by taking ordinary derivatives.

We question what will happen when we take derivatives of $t^{-\frac{1}{2}}$ because of its behavior at $t = 0$. Let us force it to behave normally; that is, let us examine the derivatives of

$$\frac{1}{g + t^{\frac{1}{2}}}\, 1$$

and then let g approach zero. The value of this expression at $t = 0$ is simply $1/g$. So, remembering, as was proved in Chapter VIII, (267), that

$$pf\,(t)\, 1 = f'(t)\, 1 + f\,(0)\, p\, 1$$

a reasonable evaluation of $pt^{-\frac{1}{2}}\, 1$ is

$$\lim_{g \to 0} p \frac{1}{g + t^{\frac{1}{2}}}\, 1 = -\frac{1}{2} t^{-\frac{3}{2}}\, 1 + \lim_{g \to 0} \frac{1}{g}\, p\, 1.$$

Using this process again there is obtained for $p^2 t^{-\frac{1}{2}}\, 1$ the expression:

$$\frac{1}{2}\frac{3}{2} t^{-\frac{5}{2}}\, 1 - \lim_{g \to 0} \frac{1}{2}\frac{1}{g}\, p\, 1 + \lim_{g \to 0} \frac{1}{g}\, p^2\, 1.$$

Thus for

$$\sum a_n p^n t^{-\frac{1}{2}}\, 1$$

there is obtained the usual series of derivatives plus an expression:

$$\lim_{g \to 0} \frac{1}{g} (a_1 p + a_2 p^2 + \cdots)\, 1$$

$$-\tfrac{1}{2} \lim_{g \to 0} \frac{1}{g} (a_2 p + a_3 p^2 + \cdots)\, 1 + \cdots. \quad (599)$$

This is much the same sort of expression as (595), when $\sum a_n p^n 1$ is examined. We should then expect that it would be legitimate to take the usual derivative under much the same conditions that apply to dropping the series of integral powers applied to the unit function. The two questions actually arise together in a problem. When making an asymptotic development both of them were settled by examining the nature of the generating function, and when considering $\epsilon^{-\beta x} 1$ the same thing was accomplished by comparison with a known result. This discussion above, however, shows why the two are bound up together and why it is not admissible to settle one without considering the other. This emphasizes the fact that when examining an operator, especially one involving positive roots, it is necessary to adhere to one scheme of treatment throughout and not swap definitions in the middle of the stream.

14. **Summary.** The series expansion of operators is thus a powerful tool which will yield a result with a minimum amount of mathematics involved in the actual transformations. There are ordinarily two ways of expanding an algebraic operator, one of which yields a convergent result, and the other, an asymptotic result. When the latter is employed, the generating function must satisfy certain criteria. Ordinarily the expansion in ascending powers of p yields this asymptotic expansion and, subject to these criteria, the integral powers in the expansion may be dropped. It is also possible for such treatment to yield a convergent result; and then the dropping of integral powers requires further justification, as by considering the operator by some other process. The separate terms in a result in the form of a series have definite physical significance, and this can be noted by obtaining such results by successive approximations. Finally, in obtaining series developments, especially where there are spurious positive poles present, it is essential that one scheme of treatment be followed throughout a given problem.

PROBLEMS ON CHAPTER XII

Prob. 1–12. What is the value of sinh $qx\ 1$ where $q = \sqrt{rcp}$?

Prob. 2–12. What is the propagation operator for a transmission line having all four constants? What three effects does it produce?

Prob. 3–12. A battery E is connected to a long transmission line of length D, open at the distant end. What is the period of the line oscillations? The period if the line is grounded? At the end of three-quarters of this period, following the battery connection to the open line, what has been the net contribution of energy to the line from the battery? Consider the line to be without losses.

Prob. 4–12. Construct the space diagrams for traveling waves of current and voltage (1) on an open line without any losses, (2) on a shorted line without any losses. In these diagrams give the position of the wave for every quarter period. Draw the time diagrams for the voltage and current at the midpoint of the line for these two cases. Repeat for the far-end terminal.

Prob. 5–12. An exponential voltage $\epsilon^{-at}\ 1$ is suddenly applied to the terminal of an infinite cable having only r and c. Obtain an expression for the entering current, in the form of an asymptotic series.

Prob. 6–12. It is desired to obtain for computation of a set of tables two series expansions of a given function $F(t)$. Show how the divergent series, if it exists, might be obtained by operational methods and the use of the infinite integral theorem.

Prob. 7–12. Find the expression for the temperature gradient at the end of a heated rod when a constant temperature source is suddenly applied there. The rod may be considered the center one of a large bundle of rods, the ends of which are all exposed to the same heat source at the same instant.

Prob. 8–12. A steam boiler is half full of water, and a large iron pipe extends for a considerable distance away from the boiler. The entire system is at room temperature. A fire is now started under the boiler, and supplies heat at a constant rate to the water. The end of the pipe dips into the water and is thus maintained at water temperature.

a. Consider only the first part of the time after starting the fire, during which it is sufficiently accurate to assume that the water temperature increases at a constant rate, and formulate in detail the procedure by which you would evaluate the distribution of temperature along the pipe as a function of the time.

b. Treat the modification you would make in this analysis if the entire procedure is considered, the boiler being vented so that the water boils without producing pressure, and a valve in the pipe is closed so that no steam enters the pipe.

CHAPTER XIII

OPERATORS INVOLVING POSITIVE ROOTS

Several times we have encountered operators of such form,

$$f(p) = \frac{1}{Z(p)} \tag{600}$$

that the impedance function Z had a root with a positive real portion. Thus, for example, we met the operational expression:

$$\frac{1}{p - b}\, 1$$

where b is a positive real quantity, and we questioned whether it is legitimate to set this equal to $\frac{1}{b}(\epsilon^{bt} - 1)$. This is not at all a simple matter, and in this present chapter the interpretations that may be given to operational expressions of this sort will be discussed.

First operators will be classified in accordance with the nature or location of the roots of the impedance function. Then the ways in which positive roots may be encountered in carrying out the operational analysis of circuits will be examined. After that the various definitions used in the interpretation of operational expressions will be treated, and the results to which they will lead if applied to operational expression involving positive roots examined. This will necessitate considerable physical discussion of non-dissipative networks. Finally working rules by which difficulty may be avoided due to operational expressions of this sort will be gathered together.

2. **Prospective and Retrospective Operators.** The generating function and the impedance function are reciprocals:

$$f(\lambda) = \frac{1}{Z(\lambda)}\,.$$

At any point (that is, at any value of λ) where $Z(\lambda)$ becomes zero, $f(\lambda)$ will become infinite. The roots of $Z(\lambda) = 0$ are hence values of λ at which $f(\lambda)$ has singularities; and if $Z(\lambda)$ is an algebraic expression, these roots are values at which $f(\lambda)$ has poles.

The generating function of an ordinary finite dissipative network has poles only on the left of the axis of imaginaries; that is, the roots have negative real portions. In fact, the expansion theorem and the infinite integral expression of Carson were derived on the assumption that all of the poles were so located. These poles correspond to natural modes of oscillation of the network with damping factors which are exponentials with negative exponents, so that the transient terms decrease as t increases. A positive pole would correspond to a current term increasing without limit; and, as was noted, this would be impossible in a finite dissipative network. It would seem, therefore, that positive poles need never be bothersome in analyzing ordinary networks. Nevertheless they can crop up in the course of solutions, and we must be prepared either to avoid them or to handle them properly.

When the generating function of an operator has all of its poles located to the left of the axis of imaginaries, such an operator is said to be **retrospective.** The reason for this name will appear below.

It is possible to set up an operator such that the poles all fall to the right of the imaginary axis, and such an operator will be called **prospective.**

When there are poles on each side of the axis, the operator is said to be **binary.** There are also operators involving poles exactly on the axis. This is a limiting case for which no special name is necessary.

The operator of Section 1 is prospective. An operator such as

$$\frac{1}{p^2 - b^2}$$

is a binary operator.

3. **Networks Involving Negative Resistance.** There are three ways in which we can encounter operators which are prospective or binary in the course of solving network problems by the operational methods with which this text deals. These will be enumerated and discussed briefly at this point, returning to them after examining the interpretations put upon prospective and binary operators by the various schemes of definition.

First, a network can involve a *bona fide* negative resistance. We shall call such a network a **cumulative** network. There is nothing physically impossible in this. Thus a series-wound dynamo driven at constant speed will generate a voltage proportional to the current through it and in the direction of that current, at least over such a range of current that saturation of the iron may be neglected. When such a generator is connected as an element in a network, it acts as a negative resistance plus, of course, some inductance. If such a machine be short circuited, the current through it will build up exponentially without limit. Physically this means that the current will increase until, due to saturation or the like, the simple linear relations no longer hold. Also there are certain arc arrangements and thermionic tube connections which, over a reasonable range, yield an effect entirely representable mathematically as a negative resistance. The operator for an element of this sort in series with an inductance is the prospective operator of Section 1. Were this subject limited to such cases we should not be greatly concerned, for it is seldom that an analysis is required of the behavior of cumulative networks, at least when they are so performing that the negative resistance is strictly representable as a constant parameter. By reason of their very nature, the performance of cumulative networks soon takes them out of the range where linear relations hold. Thus when negative resistance is employed to construct an oscillator, the amplitude of oscillation (if oscillation occurs at all) will increase until line-

arity no longer holds, for the circuit cannot remain stable at any intermediate value. The actual use of operators of cumulative networks is hence very limited, and the analysis of networks of this type belongs rather under the subject of networks with variable parameters. Yet it will be worth while to examine the treatment of prospective or binary operators which arise in this way, for it will help to fix ideas.

4. **Operational Transformations Introducing Positive Roots.** Second, prospective or binary operators can appear in the analysis of ordinary dissipative networks by reason of some operational transformation by which the retrospective operator of the network is split into parts, one of which is prospective or binary. For example, an ordinary resistance-inductance circuit has a retrospective operator of the form:

$$\frac{1}{p+b}.$$

Now, this could be written

$$(p-b)\,\frac{1}{(p^2-b^2)}$$

and the second factor treated separately. This gives a binary operator with plenty of chance for error in the analysis. Of course there is no particular point in thus making a complicated operator out of a simple one and unnecessarily getting into difficulty. There are times, however, when this same sort of thing can occur in much more subtle fashion.

Suppose that $f(p)\,1$ is being evaluated, and that this operator has arisen in considering the behavior of a dissipative network under the action of any suddenly-applied e.m.f. Now apply the shifting scheme (281), and write this:

$$f(p)\,1 = \epsilon^{-bt} f(p-b)\,\frac{p}{p-b}\,1 \qquad (601)$$

which will be abbreviated as

$$f(p)\ 1 = \epsilon^{-bt}F(p)\ 1. \tag{602}$$

Now it is possible that the generating function $F(\lambda)$ may have poles to the right of the axis of imaginaries, so $F(p)$ may be a prospective or a binary operator. If it is, we have not done something which is operationally wrong, but have very likely taken a step which is practically indiscreet.

In applying the shifting process, the assumption is made that in the result to be obtained by evaluation of $f(p)\ 1$ there is an exponential factor ϵ^{-bt}. That is, if $A(t) = f(p)\ 1$ is the desired result, then the assumption is made that $A(t)$ can be written in the form:

$$A(t) = \epsilon^{-bt}\phi(t) \tag{603}$$

where $\phi(t)$ is a function which is algebraically zero, or finite, or at most algebraically infinite at $t = \infty$. If it is, then well and good, the operator $F(p)$ will be retrospective.

But suppose our guess is wrong and $A(t)$ actually contains no exponential factor. We have artificially taken one out when there was none there. In such a case $F(p)$ must come out such as to correct our misstep, and on evaluation yield a factor ϵ^{bt} to cancel the ϵ^{-bt} we assumed present. $F(p)$ must then be prospective or binary.

Therefore, if a prospective or binary operator is encountered when introducing a shift of exponential, we may be certain that an exponential has been extracted from a result which did not contain it as a factor. There is then no real sense in using the shift, and we should be better off without it. There is thus no need of running into prospective or binary operators in this connection. But if we insist on performing the shift where a shift is not in order, we can still do so and come out with a correct final result provided the resulting prospective operator is evaluated in proper fashion.

A shift which yields a resulting operator the generating

function of which has poles along the axis of imaginaries is a border case which does not give trouble. For example suppose

$$f(p)\ 1 = \frac{p}{p+a} \tanh \frac{s}{2} (p+a)\ 1. \tag{604}$$

An exponential is shifted by the use of (281), and

$$f(p)\ 1 = \epsilon^{-at} \tanh \frac{sp}{2}\ 1. \tag{605}$$

Now the $\tanh \frac{sp}{2}$-operator involves an infinite number of roots, all on the axis of imaginaries, and none to the right of this axis. It is, in fact, the operator for the rectangular function treated in Chapter X. The operator we wished to evaluate thus gives the function of Figure 54.

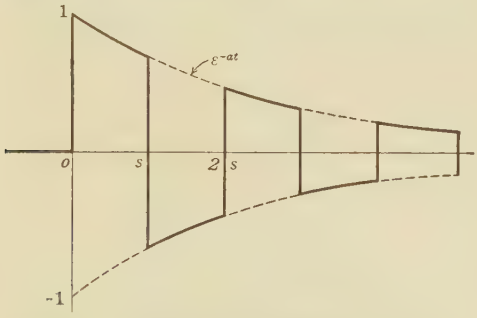

FIG. 54. Damped Rectangular Function.

As an example of an injudicious shift, consider the operational expression:

$$\frac{p\omega}{p^2+\omega^2}\ 1$$

the result of which is known to be $\sin \omega t$. Suppose we did not recognize it, but performed a shift:

$$\frac{p\omega}{p^2+\omega^2}\ 1 = \epsilon^{-\omega t} \frac{p}{(p-\omega)} \frac{(p-\omega)\omega}{(p-\omega)^2+\omega^2}\ 1$$

$$= \epsilon^{-\omega t} \frac{p\omega}{p^2 - 2p\omega + 2\omega^2}\ 1. \tag{606}$$

The generating function corresponding to this operator now has positive poles at $p = \omega(1 \pm j)$, and this is a warning that we have started on the wrong track and had better adopt another process. Yet if we insist on proceeding, we can evaluate

$$\frac{p\omega}{p^2 - 2\,p\omega + 2\,\omega^2}\,1 = \epsilon^{\omega t} \sin \omega t \tag{607}$$

by developing in a convergent series and properly gathering terms, thus arriving at a correct result after all.

5. **Ambiguity of Sign of a Radical.** Third, prospective or binary operators can be encountered in problems dealing with infinite lines, even when the network is dissipative, by reason of the ambiguity of sign of a radical. Cases of this sort have already been met in Chapters XI and XII.

This question arises even in very simple cases. Thus, for the infinite cable,

$$i_0 = E\sqrt{\frac{cp}{r}}\,1 \tag{608}$$

which gives the result:

$$i_0 = E\sqrt{\frac{c}{\pi r}}\,t^{-\frac{1}{2}}. \tag{609}$$

However this could just as well have been written

$$i_0 = -\,E\sqrt{\frac{c}{\pi r}}\,t^{-\frac{1}{2}}. \tag{610}$$

The first expression corresponds to a current directed into the cable, and the second to a current directed out of the cable against the e.m.f. of the source. The first corresponds to the actual physical conditions of the problem to be solved, while the second is a spurious result obtained in just the way that spurious results often appear in algebraic problems by reason of the double sign of a radical. The second corresponds to the solution of some problem but not to the one we had in mind. It is in fact the solution of the prob-

lem of the sudden application of an e.m.f. E to an infinite cable which is initially charged throughout to a voltage $2E$. This involves an infinite reservoir of energy to which the gates are opened by closing the switch.

In this simple problem it is easy to keep matters straight by a little physical reasoning, and it is possible to entirely ignore the fact that the radical is ambiguous in sign. Mathematics is not, however, so obliging. If an operator has double-valued properties, a mathematical treatment will very likely include this feature, perhaps when we little suspect it; and it can thus give an answer to a question we did not intend to propound.

Consider an infinite cable with a positive resistance R in series, and apply a unit e.m.f. to the combination. The operational expression is

$$i_0 = \frac{1}{R + \sqrt{\frac{r}{cp}}} 1 \tag{611}$$

that is,

$$Z(p) = R + \sqrt{\frac{r}{cp}} \tag{612}$$

and

$$Z(\lambda) = 0 \qquad \text{gives} \qquad R + \sqrt{\frac{r}{c\lambda}} = 0$$

or

$$\frac{r}{c\lambda} = R^2$$

so that there is a root at

$$\lambda = +\frac{r}{cR^2}. \tag{614}$$

Actually, of course, the network is dissipative and the current does not increase exponentially. Yet if we proceed carelessly and, for example, go through the motions of applying the expansion theorem to a case of this sort without

noting that the conditions under which it has so far been treated bar a case with positive roots, we may obtain just such an absurd result. We can solve this problem if we use operational processes which are legitimate for the purpose in hand, but we cannot use any formula we meet under any conditions without getting into trouble.

The positive roots that occur in connection with the operators of infinite lines are spurious and do not correspond to the physical problem of the sudden application of an e.m.f. to an uncharged network. We hence wish to avoid them by adhering to one set of signs throughout the treatment. This is none too easy to accomplish at times.

6. **Finite Lines.** In the case of finite lines, with or without various terminal arrangements, the difficulty of sign ambiguity does not arise.

The alternating-current impedance of a finite line with all four constants present is

$$Z(j\omega) = \frac{\sqrt{r + jl\omega}}{\sqrt{g + jc\omega}} \tanh\left(D\sqrt{r + jl\omega}\sqrt{g + jc\omega} + \tanh^{-1}\frac{\sigma(j\omega)\sqrt{g + jc\omega}}{\sqrt{r + jl\omega}} \right) \tag{615}$$

where the length is D, the line being terminated at the distant end by an impedance $\sigma(j\omega)$. The impedance operator is obtained by replacing $j\omega$ with p. There is no ambiguity here. If the sign of either of the radicals is changed, say $\sqrt{r + lp}$, the sign or value of the complete expression is unaltered. It is hence amenable to ordinary treatment, and no poles to the right of the axis can arise to disturb our peace of mind. The same is true of course with operators for finite cables, or with other expressions obtained from the above. In reasonable physical problems on dissipative networks, therefore, there is little need of running into spurious poles on the right; and if they do arise by reason

of the use of the convenient mathematical fiction of an infinite line, the problem can still be solved correctly.

7. **Importance of Definitions.** We have emphasized several times that the interpretation of an operational expression is more or less arbitrary and depends upon the definitions that we choose to set up for its evaluation.

An operational expression:

$$f(p)\,1$$

for a given dissipative network problem merely sums up in shorthand form the information that is contained in the differential equations of the network plus a certain set of initial conditions, namely, that the network is initially in equilibrium and that a unit e.m.f. is suddenly applied to it. This expression is then interpreted to obtain a resulting $A(t)$ in the form of a time function. This is done by comparison with some mathematical scheme of solving networks of this sort, such as the Fourier integral treatment, and thus a general definition is set up by which the operational expression can be evaluated. But this can be done in several ways, for there are various methods of attacking network problems by means of classical mathematics. So there may be several distinct definitions in accordance with which an operational expression can be interpreted.

These definitions have one thing in common. When applied to an operational expression representing the response of a finite dissipative network to unit e.m.f. they will all yield the same result, for they were set up to do that very thing. But an operational expression written at random, say

$$\log \sin p\,1$$

for example, has no immediate physical meaning whatever and is not related to a finite dissipative network. If such a random operational expression is interpreted by various

definitions, there will be obtained — as is to be expected — various discordant results. The definitions will yield identical results only on operational expressions which fall within the limitations by which they were all bound in their derivation.

Moreover, most of the definitions were set up with at least the implied reservation that the generating function concerned had no poles to the right of the axis of imaginaries. It has been seen that such poles can occur even when dealing with dissipative networks. In interpreting operational expressions involving positive poles, even though spurious, we may well expect to obtain different results on using different definitions. It is necessary, therefore, to review the definitions and determine which ones can be depended upon to give the correct answer to any given problem involving positive poles arising either spuriously or by reason of the treatment of an actual cumulative network. In other words, having definitions at hand which work properly when positive poles are barred, it is necessary to find out what these definitions will yield in the way of results when such poles do occur, and which of these results correspond to physically important problems. There is no *à priori* reason however, for expecting them to correspond to interesting physical problems at all.

8. **An Example. Two Questions Concerning the Performance of a Cumulative Network.** Before entering upon this matter of definitions in general it will be advisable to discuss further the very simple example of Sec. 1 — a negative resistance in series with an inductance — to illustrate how different results can arise from the same prospective operator.

Examine the problem first by means of a series expansion. Write the operational expression:

$$\frac{1}{p-b}1 = \left(\frac{1}{p} + \frac{b}{p^2} + \frac{b^2}{p^3} + \cdots\right)1 \qquad (616)$$

which we consider term by term, obtaining,

$$\left(t + \frac{bt^2}{2!} + \frac{b^2t^3}{3!} + \cdots\right)1$$

or

$$\frac{1}{b}(\epsilon^{bt} - 1)\,1$$

which is plotted in Fig. 55. This is in accordance with Volterra's teatment. A result is obtained which is zero until $t = 0$, and which then increases exponentially. This result is evidently the answer to a question concerning a physical problem. It is: Given a circuit containing inductance and negative resistance, open circuited until time $t = 0$ when it is closed on a source of unit e.m.f., what will be the progress of the current thereafter?

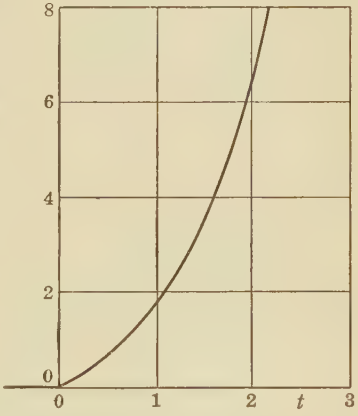

FIG. 55. The Function $\frac{1}{b}(\epsilon^{bt} - 1)1$, with $b = 1$.

Now let us treat the same problem by the expansion theorem, or rather take as a starting point the expression used by Wagner for the unit function,

$$1 = \frac{1}{2\pi j}\int \frac{\epsilon^{\lambda t}}{\lambda} d\lambda \qquad (617)$$

the path of integration being along the axis of imaginaries, avoiding the pole at the origin by a small half circle to the right. The response of a network for which the operator is $f(p)$ is then

$$f(p)\,1 = \frac{1}{2\pi j}\int f(\lambda)\frac{\epsilon^{\lambda t}}{\lambda} d\lambda \qquad (618)$$

taken over the same path. This may be taken as one definition for the interpretation of an operational expression. It is, in fact, the definition used by Wagner and Wiener. Applied to the expression of the simple example above, it becomes

$$\frac{1}{p-b}1 = \frac{1}{2\pi j}\int \frac{\epsilon^{\lambda t}}{\lambda(\lambda - b)}\,d\lambda. \tag{619}$$

Evaluate this integral. For $t < 0$ the path of integration can be closed by a large half circle to the right. But the value of the integral is not then zero for the contour encloses a pole at the point $\lambda = b$. Its value is, in fact, $2\pi j$ times the residue at the pole, which is

$$\lim_{\lambda \to b} \frac{(\lambda - b)\epsilon^{\lambda t}}{\lambda(\lambda - b)} = -\frac{1}{b}\epsilon^{bt} \tag{620}$$

where the minus sign appears because the pole was encircled in a negative direction. When $t > 0$ the path may be closed by a large half circle to the left. The contour encloses only the pole at the origin, and the value of the residue is

$$\lim_{\lambda \to 0} \frac{\lambda\epsilon^{\lambda t}}{\lambda(\lambda - b)} = -\frac{1}{b}. \tag{621}$$

The evaluation of the operational expression in accordance with this definition thus gives

$$\begin{aligned} \frac{1}{p-b}1 &= -\frac{1}{b}\epsilon^{bt} \qquad t < 0 \\ &= -\frac{1}{b} \qquad t > 0 \end{aligned} \tag{622}$$

or as it may be written,

$$\frac{1}{p-b}1 = \frac{1}{b}(\epsilon^{bt} - 1)\,1 - \frac{1}{b}\epsilon^{bt} \tag{623}$$

which is plotted in Fig. 56. This is an utterly different result from the one previously obtained. It contains an extra term, and this corresponds to a current existing before $t = 0$.

The current no longer increases exponentially when $t > 0$, but it is constant from $t = 0$ on. A new interpretation of an operational expression has been obtained, and we inquire whether we have again found an answer to a question concerning a physical problem. Contrary to the usual condition we have a solution first, and seek the problem

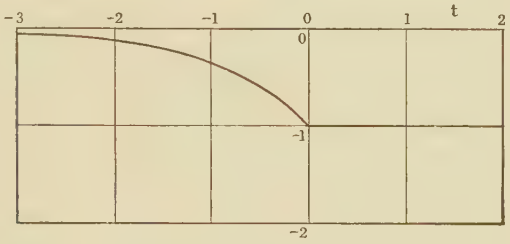

FIG. 56. The Function $\frac{1}{b}(\epsilon^{bt} - 1)1 - \frac{1}{b}\epsilon^{bt}$, with $b = 1$.

afterward. The definition has interpreted the operational expression into a result, but a result for what physical conditions?

Before answering this let us digress slightly and consider the ways in which an e.m.f. can be suddenly applied to a circuit. There are two ways in which it can be done. The circuit can be open initially, and then suddenly closed on the source of e.m.f.; or it can have been closed indefinitely and the e.m.f. then introduced suddenly into this closed circuit. This latter process might be visualized thus: We have in a closed circuit, without other e.m.f.'s, an unexcited, direct-current generator which is running at constant speed, and we suddenly close its field circuit on a separate source of excitation. In a physical machine, of course, the field is not established instantly, but if this lack of abruptness is neglected we have a picture of the process. With a dissipative circuit it makes utterly no difference which way of introducing the e.m.f. is involved. With a dissipative, stable, dead circuit having positive resistance nothing hap-

pens anyway until the e.m.f. is introduced, whether the circuit be closed or open. But with a cumulative, unstable, live circuit of negative resistance a great deal can happen even when there is no applied e.m.f.; and when such circuits are involved it is hence imperative that careful distinction be made between the two ways in which the e.m.f. can be introduced.

It should be emphasized that there is nothing mystical about the operational method which automatically takes care of a question like this. The operational method solves what it is told to solve, and its instructions are contained in the definitions on which it is based. In this matter, the instructions have heretofore not been given explicitly for the simple reason that it was immaterial which case was set up when dealing with dissipative networks. But the definitions cover the point implicitly, and the various operational processes must be carefully examined before it is possible to tell what problem they are solving.

The same dilemma can perhaps be visualized more clearly if we discuss for a moment an unstable mechanical system. Consider a pencil standing on its point on a table, and a unit force in a horizontal direction suddenly applied to the pencil's top. Evidently the whole problem is not yet clearly stated, and other conditions are to be inserted. One way of completing the problem is as follows: If the pencil is held vertical until $t = 0$, and then released and the force applied at the same instant, in what manner will it gather speed as it tips over? Corresponding to this there is a similar question which might be asked for the cumulative network above when the e.m.f. is inserted in the first way. But suppose the pencil were not fixed vertically, but were left to itself until $t = 0$. Evidently it would be improper to ask: The pencil having been balanced for an indefinite period, what will happen when the force is applied? For the pencil will not remain balanced for an indefinite period. However a question can be propounded thus: If the pencil

is tipping over at $t = 0$ when the force is applied which just arrests its motion, how far must it have tipped prior to $t = 0$ in order that this can be possible? There is to this, likewise, a corresponding question which might be asked for the cumulative network above into which the e.m.f. is inserted in the second way. It is a fantastic question to be sure, but it happens to be one which corresponds to a method of treatment of an operational expression.

There are then two principal questions which can be asked regarding a cumulative network. The first is simple. The network being open at all points, and then closed at $t = 0$ with the inclusion of an e.m.f., what will happen?

The second is not so simple. Shall it be: The network having always been closed and the e.m.f. introduced at $t = 0$, what will happen? Such a question is improper. If an unstable network has been left closed from $t = -\infty$ on — or even say for a few years — although there has been no applied e.m.f. present a great deal will have happened before it is inserted. There will be no chance to perform the experiment. The current will have grown somewhere beyond the range of linearity — or more likely have blown a fuse. The experiment will have been concluded and the circuit blown up before it can be operated on. Currents and voltages in cumulative networks tend to become exponentially infinite. Moreover, the exact way in which the circuit will have blown up will be utterly indeterminate, depending as it does mathematically upon infinitesimal departures from quiescence occurring in the infinitely remote past. Our question, like other questions that are sometimes met with by students, and witnesses in court, is improper and out of order.

With respect then to a cumulative network, what question can be asked that is proper within the premises? Roughly it is: How can such a circuit be controlled so that it will not blow up? More explicitly it is the following: The circuit having been closed, the e.m.f. introduced at

$t = 0$, and the currents and voltages remaining finite or at least not becoming exponentially infinite thereafter, what must have been the past history of the circuit? In other language, if an unstable network is building up exponentially and going out of control, at what part of the growth can a given e.m.f. be inserted to avert a catastrophe? This then is the second question that may be properly asked concerning a cumulative network.

It is evident in the example treated above that the definition based on Volterra's treatment gave an answer to the first question, and the definition based on Wagner's treatment gave an answer to the second question. We are not now concerned with whether the answer to either of these is useful. Rather we have a considerable amount of operational machinery already set up, and are attempting to find out what sort of product it will yield if it is set to work on an operator involving positive roots.

It has now been seen that there are several classes of operators, named in accordance with the location of the poles. There are two sorts of networks, dissipative and cumulative. The latter always have operators which involve poles to the right of the axis of imaginaries, but the former can also have operators which involve such poles. It has also been observed that the result given by an operational expression involving poles to the right of the axis depends upon the definition under which it is interpreted. Concerning cumulative networks subjected to suddenly applied e.m.f.'s, it has been noted that there are two questions which may be properly asked, and that the interpretation of an operational expression will give an answer to one of these when the interpretation is made by one definition, and to the other when the interpretation is made by a second definition. We shall now digress again for a moment on the subject of the names of the classes of operators, which is a matter which might otherwise cause confusion. Then the various definitions under which operational ex-

pressions can be interpreted will be examined, and the matter of their consistent use when positive poles are involved treated.

9. **Names of the Classes of Operators.** The reason why the classes of operators are termed retrospective and prospective can now be better understood. These names imply that the first class in some way involves the characteristic of looking backward, and the second of looking forward. The first involves something which is historical, and the second something which is prophetic. When an operational expression involving a retrospective operator is evaluated, the past history of the network is fixed, the condition surrounding the solution being that nothing shall have happened in the network **before** the time the e.m.f. is introduced. The operational result is thus determined by a known past; and the operator which yields its result in view of this past is called retrospective. On the other hand, when an operational expression involving a prospective operator is evaluated by a method which yields an answer to the second question regarding a cumulative network, it is the future of the network performance that is fixed. In this case the condition surrounding the solution is that nothing (nothing violent, at least) shall be allowed to happen in the network **after** the e.m.f. is introduced. The operational result is thus determined by a known future; and the operator which yields its result in view of this future is called prospective. It has been shown however, that this same prospective operator when evaluated by some other method may yield a result which corresponds to the first question rather than to the second. To some extent, therefore, the names are not entirely descriptive. The real distinction between the classes of operators is the location of the poles; and the names are probably as good as any for the purpose of denoting this distinction, especially as they have been used previously in operational literature for this purpose. In this previous literature the names were adopted with a

single definition in mind. Here in this text where various definitions are brought together, it is seen that a generating function plus a definition is necessary to determine the result of an operator. The names cover the behavior of the generating function only. Binary, of course, means simply partaking of the nature of both kinds, or with poles on both sides of the axis.

When a generating function has poles on the axis of imaginaries it does not fall within any of our classifications. It is not necessary to have special names for this purpose, for the limiting case of poles exactly on the axis does not give any especial difficulty. Two poles at a pair of pure imaginary points correspond to a sustained oscillation.

10. **The Result of Employing Various Definitions.** We can now consider some of the various definitions and note what they will yield when used to interpret operators which are prospective or binary.

Volterra's treatment proceeds on the premise that an operator applied to a time function which is zero when $t < 0$ will yield a result which is also zero when $t < 0$. When series expansions of operators are used to yield convergent series in t, and Volterra is followed, a problem is treated in which the network carries no current until the time when the e.m.f. is inserted. This will answer the first question for a cumulative network. If an operator involving a spurious positive root is handled by this process, we should expect to obtain a true result for a dissipative network if we make proper choice of the sign of the radicals which appear.

Wiener's treatment is based on the behavior of the generating function along the axis of imaginaries. Applied to a cumulative network it will answer the second question. However, the evaluation is performed in a single sheet of a Riemann surface[1] and spurious poles which enter by reason of sign ambiguity are avoided. Hence when used to obtain an asymptotic result for a dissipative network it yields a

[1] See footnote, p. 150.

true result in spite of any such ambiguity which may be present. In Wiener's treatment it is postulated that the result of an operator shall not be exponentially infinite in either direction along the axis of reals. It is therefore bound to yield an answer to the second question concerning a cumulative network.

In deriving the expansion theorem it was assumed that there were no poles to the right of the axis of imaginaries. The treatment can still be carried through when such poles exist, but in several ways. Hence when such poles are present it is possible to have different forms of the expansion theorem which will yield results consistent with various other schemes of treatment. In Sec. 8 above we employed on a particular operator a form of the theorem which was in accordance with the definition of that operational expression in terms of an integral along the axis of imaginaries. Terms corresponding to the positive poles were included, but only when $t < 0$. When $t > 0$ there were included only terms corresponding to poles to the left of the axis. This treatment was seen to yield a result which could answer the second question, and to be consistent with the definition on which Wiener's treatment is based. It was therefore perfectly proper in Chapter XII, Sec. 12, to treat one part of an operator in accordance with Wiener's development into an asymptotic series, and the other part by this special form of the expansion theorem.

We may, however, derive the expansion theorem as was done in Chapter VII, following Heaviside, and when we come to the interpretation of $1/(p - b)$ adopt the first procedure of Sec. 8 of this present chapter, thus following Volterra. The theorem will then be derived in just the form there written, but terms will be obtained corresponding to all of the poles, those on the right of the axis included. The expansion theorem with this interpretation thus yields results which answers the first question, and may be consistently used in the analysis of any problem where

the parts of operators are being interpreted by definitions consistent with that of Volterra.

Bromwich early set up a definition in terms of a line integral in the complex plane, and this is reviewed and utilized by Jeffreys. The path chosen by Bromwich however, is such as to pass to the right of all poles of the generating function. Hence applied to a cumulative network it yields an answer to the first question. It also avoids all difficulty due to spurious poles, although the mathematical difficulties involved in its use may be considerable. It leads to the expansion theorem with terms corresponding to all poles included. Bromwich's and Volterra's processes are thus consistent.

In considering the Fourier integral treatment the discussion was limited to the condition that $A(t)$ be zero for t negative. It was also required that the behavior of $A(t)$ for large positive values of t be such that the infinite integrals converge. The Fourier transform effectively utilized by Carson is similarly limited, for it is a part of the Fourier analysis. Definitions based on Fourier analysis cannot therefore be used to answer either of the questions regarding cumulative networks, although it would be possible to extend this treatment to yield an answer to the second question if that were desirable. Thus when we encounter positive poles associated with an operational expression, we cannot expect to obtain a proper interpretation by means of these definitions.

Operators involving poles on the axis of imaginaries constitute a limiting case. When interpreted by definitions in terms of a line integral along the axis of imaginaries it is necessary to avoid these poles by small half circles. If these are taken to the right of the axis, the result obtained will be the same as though the interpretation were by Bromwich's definition. Interpretation by Fourier integral methods is also possible, and the results will be consistent with these other definitions.

There are thus two main schemes of definition. In the first are those of Volterra and Bromwich; and on cumulative network problems they yield results which correspond to the first question that was propounded. In the second are those of Wiener and Wagner; and they give results corresponding to the second question. If in any problem involving positive poles a single scheme is followed, consistent results will be obtained. If the network is dissipative, such a consistent treatment will yield correct results, even if spurious positive poles are encountered, and the answers to the two questions are then identical.

It is possible to use other definitions than those which have been reviewed, but the above are the principal ones with which we are concerned.

11. **Working Rules. The Avoidance of Inconsistency.** We are now in a position to set up working rules which will avoid the chance of error in connection with prospective or binary operators.

It is understood that such operators can appear in three ways: In the treatment of cumulative networks, by reason of indiscreet operational transformations, and spuriously by reason of ambiguity of sign.

If we insist on treating the operator of a cumulative network operationally it is necessary to first specify exactly the physical problem to which we seek the solution; that is, it is necessary to state whether the network is to be considered open or closed prior to $t = 0$. If the former, then we can usually solve by a series expansion to obtain a convergent result, or we can utilize the definition of Bromwich. The latter problem will hardly occur practically, but it can be treated by a modified expansion theorem, or by following Wiener. The Fourier integral treatment cannot be used in the form in which it is developed here.

When we obtain positive poles as a result of operational transformations on a dissipative network we are not required to handle the resulting prospective or binary oper-

ator. Rather we should refrain from the transformation which introduces them, for it is worse than useless.

Finally, note that positive poles appear by reason of ambiguity of sign only when treating infinite lines or cables; and the ambiguity can be avoided by treating finite systems, although at the cost of much complication. The series development of the operator to obtain either a convergent or an asymptotic development does not become involved with this question if the operator as a whole can be thus treated. Often, as was the case when the infinite cable was first treated, the entire difficulty is removed by carefully retaining only one sign for the radical. Unfortunately this question of sign ambiguity is likely to become involved with other matters which are also difficult, as was the case in the illustrative problem of Sec. 10 of the preceding chapter.

When an operator has spurious poles to the right of the axis of imaginaries by reason of the presence of a radical, the prime requisite is that one definition be followed throughout in making the interpretation. If the system is actually dissipative we are then bound to come out with the correct result; for all definitions, properly handled, lead to consistent results when treating dissipative systems. On the other hand, by splitting the operator into parts and applying different definitions to each part we can arrive at entirely absurd results. To illustrate this we shall review the problem of Chapter XII, Sec. 11 dealing with a resistance in series with an infinite cable.

First review that treatment and note how the difficulty was avoided. An expansion was made in descending powers of p which led to a convergent series result. Volterra was then followed. The operator appeared thus:

$$\frac{1}{R}\left[1+\frac{1}{hp}+\frac{1}{(hp)^2}+\cdots\right] - \sqrt{\frac{c}{r}}\left[\frac{1}{hp}+\frac{1}{(hp)^2}+\cdots\right]p^{\frac{1}{2}}.$$

The second term involves a radical which might conceivably have been prefaced by a $\pm$ sign. However one of these signs was discarded because it was realized that it did not belong to the problem being considered; and the ambiguity was thus avoided by the simple process of retaining the correct sign for the radical throughout. The two terms above may be written

$$\frac{1}{R}\frac{hp}{(hp-1)} - \sqrt{\frac{c}{r}}\frac{1}{(hp-1)}p^{\frac{1}{2}}.$$

Notice that the original operator has been split into two parts, each of which is prospective. The result of applying these operators to the unit function to obtain

$$i_0(t) = \frac{1}{R}\epsilon^{\frac{t}{h}} - \frac{2}{R}\sqrt{\frac{t}{\pi h}}\left[1 + \frac{2}{3}\left(\frac{t}{h}\right) + \frac{2^2}{3\cdot 5}\left(\frac{t}{h}\right)^2 + \cdots\right] \tag{624}$$

shows this, for each part increases exponentially. However the total does not so increase, for there is a cancellation which leaves a result that tapers off. This cancellation cannot be performed formally; but it appears, when calculations are made, that the two terms approach each other. This is an excellent example of the fact that no harm is done in splitting an operator into parts that are prospective or binary, provided each part is consistently treated. There is no difficulty in ambiguity of sign here; for if the sign of the second term of the result is written as plus it is immediately evident that there is being considered, not the problem to be solved, but a problem of a cumulative network, and the exponentially-increasing result is to be expected.

An asymptotic result was also obtained following Wiener. We expanded in ascending powers of p, and two series again appeared corresponding to the two fractions above. Then, however, the first fraction was entirely discarded except for a constant term on the basis that the asymptotic expan-

sion of the rest of it was zero; and a correct result was obtained by adding this constant term to the asymptotic expansion obtained from the second term. Are we in error when we thus place the result of a fractional operator equal to $\epsilon^{\frac{t}{h}}$ in one treatment and then immediately set it equal to unity in another treatment? Not at all. Both results are correct under the respective definitions that are being used, and both lead to correct answers when the same definitions are consistently applied to the other term of the problem. But if we use the result for the first term from the Volterra definition together with the result for the second term from the Wiener definition, we shall be grossly in error as is immediately apparent. We shall obtain an exponentially-increasing current which cannot occur in a dissipative network.

It has been seen above that the value of an operational expression is given by the behavior of its generating function, considered as a function of a complex variable. Moreover it is sufficient that there be given only the behavior of the generating function along the axis of imaginaries. When there is a radical in the generating function there may result two entirely different sets of values for the generating function corresponding to the values of the argument along the axis of imaginaries, depending upon the sign that is chosen for the radical. The complete specification of an operator of this sort requires, therefore, not only the mathematical form of the generating function, but also a supplementary statement as to the signs that are to be used with the radicals. In the language of functions of a complex variable, a generating function of this sort is a multivalued function and requires for its description a two-sheeted Riemann surface. In order to specify an operator, therefore, it is necessary to specify not only the form of the generating function but also which sheet of the Riemann surface is to be considered. Any definition of an operational expression in terms of a line integral in a complex plane then becomes

unambiguous when it is stated in which sheet the path of integration lies. The poles of the function may lie on either sheet. If the spurious poles lie on the lower sheet and the definition is given in terms of a line integral in the upper sheet, difficulty from the presence of spurious poles is avoided. When the function is multivalued, a definition in terms of a line integral is evaluated only with great difficulty by the process of closing the path of integration, for there are branch points present and the contour will either pass into both sheets or surround a branch cut, both of which conditions introduce difficulties. However, a process such as Wiener's which evaluates by a treatment which is confined to a single sheet of the surface does not encounter this difficulty, and the spurious poles are entirely ignored, as they should be.

It is thus a fallacy to consider that to every operational expression corresponds one and only one correct evaluation as a time function. When there are poles to the right of the axis of imaginaries, whether or not these are spurious, the result of a given operational expression depends upon the definition by which it is interpreted. There is no harm in this. The true final result will be obtained in every case provided the network is actually dissipative, or if cumulative, that we recognize the problem being solved, and provided always that the treatment is kept consistent.

PROBLEMS ON CHAPTER XIII

Prob. 1–13. In the network of Fig. 6 assume that the source $e(t)$ consists of a series dynamo which supplies a constant voltage of one volt plus an effect represented by a negative resistance of 10 ohms. The other constants are as given in Prob. 3–7. Find the variation of current after the switch is closed.

Prob. 2–13. Write the operational expression for a cable of length D terminated at the distant end in a resistance R. Show that no ambiguity of sign will arise in treating this operator.

Prob. 3–13. Negative resistances have been seen to have a definite physical meaning. Is it possible to encounter also a physical problem involving negative inductances or capacitances?

Prob. 4–13. Apply the shifting formula to the operational expression $p^{\frac{1}{2}}\,1$, and show that a positive root appears. Treat the resulting operational expression consistently, and thus obtain the usual result.

CHAPTER XIV

SPECIAL CIRCUIT PROBLEMS AND ARTIFICES

The operational attack is not limited to the case of a suddenly-applied electromotive force. The processes may be utilized wherever two variables can be connected by an operational expression of the type considered in the foregoing treatment. Thus we may deal, for example, with the case of **suddenly-applied currents.**

As a simple example consider that a constant current I is suddenly introduced into a network as in Fig. 57. In this diagram consider G to be a constant-current generator which is introduced into the circuit by opening the switch across it at time $t = 0$. Suppose the expression for the voltage e_L across the inductance L_2 is desired. The current I divides between the two branches, and the fraction flowing through L_2 is

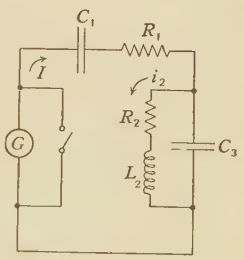

FIG. 57. Constant Current Suddenly Applied to a Network.

$$i_2 = \frac{\dfrac{1}{C_3 p}}{R_2 + L_2 p + \dfrac{1}{C_3 p}} I\,1 = \frac{1}{1 + R_2 C_3 p + L_2 C_3 p^2} I\,1. \quad (625)$$

The voltage across the inductance is then

$$e_L = L_2 p i_2 = \frac{L_2 p}{1 + R_2 C_3 p + L_2 C_3 p^2} I\,1. \quad (626)$$

We have an operational formula:

$$\frac{p\omega}{(p + \alpha)^2 + \omega^2} 1 = \epsilon^{-\alpha t} \sin \omega t \quad (627)$$

so we complete the square of our denominator:

$$e_L = \frac{1}{C_3\sqrt{\frac{1}{L_2C_3} - \frac{R_2^2}{4L_2^2}}} \frac{p\sqrt{\frac{1}{L_2C_3} - \frac{R_2^2}{4L_2^2}}}{\left(p + \frac{R_2}{2L_2}\right)^2 + \left(\frac{1}{L_2C_3} - \frac{R_2^2}{4L_2^2}\right)} I\,1 \tag{628}$$

and the result follows:

$$e_L = \frac{1}{C_3\sqrt{\frac{1}{L_2C_3} - \frac{R_2^2}{4L_2^2}}} \epsilon^{-\frac{R_2 t}{2L_2}} \sin\sqrt{\frac{1}{L_2C_3} - \frac{R_2^2}{4L_2^2}}\, t. \tag{629}$$

Such problems as these arise often in mechanics. Consider, for example, the scheme of Fig. 58 which represents a flywheel loose on a shaft and coupled to it by a helical spring.

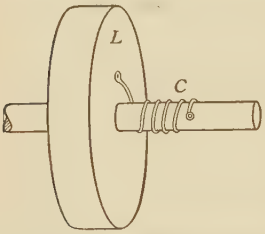

FIG. 58. Mechanical Oscillatory Circuit.

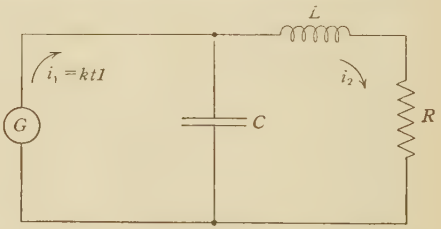

FIG. 59. Sudden Application of a Linearly-increasing Current.

Suppose we are to start the shaft, give it a constant acceleration, and find the flywheel speed as a function of time. The analogous electric circuit is shown in Fig. 59, where the generator is to furnish a current $kt\,1$, and where the current in the inductance is desired. The relation of analogous parameters is readily set up. Thus,

$$\begin{aligned} i_2 &= \frac{\frac{1}{Cp}}{\frac{1}{Cp} + R + Lp} i_1 \\ &= k\frac{1}{LCp^2 + RCp + 1} t\,1 \\ &= \frac{k}{Lp^2}\frac{Lp}{LCp^2 + RCp + 1} 1 \end{aligned} \tag{630}$$

and referring to the preceding problem, this is

$$i_2 = \frac{k}{\omega C L p^2}\,\epsilon^{-\alpha t}\sin \omega t\, 1 \tag{631}$$

where

$$\alpha = \frac{R}{2L} \qquad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

or

$$i_2 = \frac{k}{CLp(\alpha^2+\omega^2)}\left(1 - \frac{\alpha}{\omega}\,\epsilon^{-\alpha t}\sin \omega t - \epsilon^{-\alpha t}\cos \omega t\right) 1 \tag{632}$$

$$i_2 = \frac{k}{CL(\alpha^2+\omega^2)}\left\{t + \frac{1}{\alpha^2+\omega^2}\left[\left(\frac{\alpha^2-\omega^2}{\omega}\right)\epsilon^{-\alpha t}\sin \omega t + 2\,\alpha\epsilon^{-\alpha t}\cos \omega t - 2\alpha\right]\right\}. \tag{633}$$

Here i_2 corresponds to the flywheel speed, and pi_2 to its acceleration.

In heat-flow problems cases are met where the electric-circuit analogy involves a suddenly-impressed current. Consider, for example, a heating element consisting of a mica sheet wound with resistance wire, clamped between two thick plates of metal. If the heat flow is substantially perpendicular to the faces of the plates, we can consider the electric-circuit analogy of a source of current connected to two cables in parallel, as in Fig. 60. Suppose the heating ele-

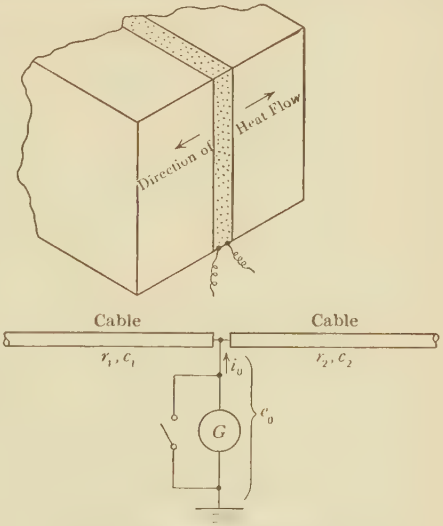

FIG. 60. Heat-flow Problem and its Electrical Analogue.

ment is turned on at $t = 0$ and is supplied with a constant number of watts, that is, a constant number of calories per second. This corresponds to a constant-current generator G in the electric problem. If heat loss from the outer surface of the blocks is negligible, this corresponds to finite cables open at the distant end. If the blocks are thick — or in any case for a short length of time, perhaps a minute — it will be sufficiently accurate to consider infinite cables. The electric circuit is formulated as below, the analogous parameters being readily noted. The home-end voltage is analogous to the temperature at the inner faces of the blocks. The impedances of the two cables are

$$\sqrt{\frac{r_1}{c_1 p}} \quad \text{and} \quad \sqrt{\frac{r_2}{c_2 p}}$$

so their admittance in parallel is

$$\sqrt{\frac{c_1 p}{r_1}} + \sqrt{\frac{c_2 p}{r_2}}$$

and

$$e_0 = \frac{1}{\sqrt{\frac{c_1 p}{r_1}} + \sqrt{\frac{c_2 p}{r_2}}} i_0. \tag{634}$$

Now since $i_0 = I_0 1$,

$$e_0 = I_0 \frac{1}{\sqrt{\frac{c_1}{r_1}} + \sqrt{\frac{c_2}{r_2}}} p^{-\frac{1}{2}} 1 = \frac{1}{\sqrt{\pi}} \frac{2 I_0 t^{\frac{1}{2}}}{\sqrt{\frac{c_1}{r_1}} + \sqrt{\frac{c_2}{r_2}}}. \tag{635}$$

The mid-temperature thus rises without limit. This will of course be modified on considering finite blocks. If the heat capacity of the heating element itself is considered, and if there is a thermal resistance between the element and the blocks which must be considered, the analogous circuit becomes that of Fig. 61. If the two blocks have the same thermal resistivity, density, and specific heat, the operational

equation is

$$e_0 = I_0 \frac{R + \sqrt{\frac{r}{cp}}}{2 + RCp + C\sqrt{\frac{rp}{c}}} 1. \qquad (636)$$

This is rather complicated but can be evaluated as a series.

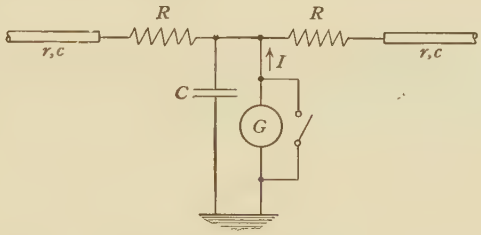

FIG. 61. Electrical Analogue of a Heat-flow Problem.

2. **Condenser Discharge through a Network.** A **suddenly-released charge** can be treated by our operational scheme. Consider a mechanical pendulum, the support of which is suddenly displaced horizontally through a small distance. This is analogous to the circuit of Fig. 62, in which a charge Q is suddenly introduced onto the condenser. The equations of the circuit are

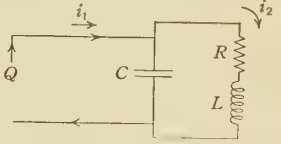

FIG. 62. A Charge Suddenly Placed on a Condenser.

$$(R + Lp)i_2 = e_C$$

$$\frac{1}{Cp}(i_1 - i_2) = e_C$$

$$\frac{1}{p} i_1 = q_1(t) = Q\, 1.$$

So

$$\left(R + Lp + \frac{1}{Cp}\right) i_2 = \frac{1}{C} q_1(t) = \frac{1}{C} Q\, 1 \tag{637}$$

or

$$i_2 = \frac{Q}{RC + LCp + \dfrac{1}{p}}\, 1 \tag{638}$$

and the solution follows as before. This is one way of treating a condenser initially charged and then connected to a network.

Another way is as follows: If a battery E be suddenly connected in series with an uncharged condenser, as in Fig. 63, the current resulting is the same as when the condenser is initially charged to voltage E and then discharged through

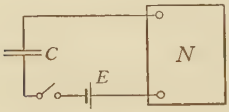

FIG. 63

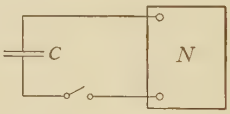

FIG. 64.

the network, as in Fig. 64, the direction of the voltage being maintained the same in the two cases. This follows because in each case the voltage $e(t)$ impressed on network N is initially E and is decreased by an amount $\frac{1}{C}\int_0^t i\, dt$ as the current flows. Since the voltage impressed on N is the same in each instance, the same current will flow, and one may be analyzed in place of the other. This enables us to reduce the problem of an initially-charged condenser to that of a suddenly-applied unit electromotive force. Consider that the condenser voltage is $\frac{Q}{C} = E$, and the impedance of N is $Z_n(p)$. Then

$$e(t) = E\, 1 - \frac{1}{Cp} i = Z_n(p) i$$

from which

$$i = \frac{E}{Z_n(p) + \frac{1}{Cp}} 1. \tag{639}$$

By this procedure the current in the problem above can be found to be

$$i_2 = \frac{E}{R + Lp + \frac{1}{Cp}} 1 = \frac{Q}{C} \frac{1}{R + Lp + \frac{1}{Cp}} 1$$

as before.

When a voltage or current is suddenly applied to a network containing an initially-charged condenser, the total effect can be obtained by superposing individual results. Thus,

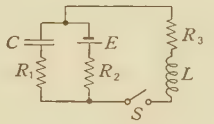

FIG. 65. Network Involving an Initially-charged Condenser.

suppose the current through L is considered when the switch of Fig. 65 is closed. One component of current is found by analyzing Fig. 66, the condenser being initially uncharged.

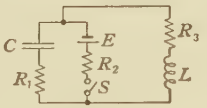

FIG. 66.

This is

$$i_1 = \frac{E}{R_3 + Lp + \frac{R_2\left(R_1 + R_3 + Lp + \frac{1}{Cp}\right)}{R_1 + \frac{1}{Cp}}} 1. \tag{640}$$

Next the second component is obtained from Fig. 67.

$$i_2 = \frac{ER_2}{\left(R_1 + \frac{1}{Cp}\right)(R_2 + R_3 + Lp) + R_2(R_3 + Lp)} 1. \quad (641)$$

The net current is now obtained by adding these two components. They may as well be combined before evaluating the operator.

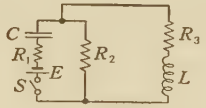

FIG. 67

3. **Discharge of a Line. Lightning Surges.** The same artifice will work on distributed systems to reduce problems of an initial energy storage in a condenser to the problem of a unit applied electromotive force. Suppose it is desired to treat the problem of the spreading of voltage and current on an aerial transmission line due to charges induced and released by lightning. This is an important practical problem, for the whole science of the proper protection of electrical apparatus against destruction by **lightning surges** depends upon a proper knowledge of how these surges are propagated.

The procedure by which such surges are set up, except in the case of direct strokes against which it is somewhat hopeless to try to protect, is about as follows: A charged cloud gradually accumulates over a portion of the line, and induces a bound charge on the line conductors, this charge slowly accumulating there through slow leakage onto the line through the terminal apparatus and over the insulators. The cloud discharges and these line charges become free, suddenly producing potentials of several hundred thousand volts above ground on a limited portion of line. It is our problem to study how the corresponding energy spreads out along the line.

At first consider for simplicity that the potential E is induced over a distance located at one end of the line, and that this end is open and the other end grounded. Other cases may be treated readily by extending the method. It may be that steady-state alternating potentials and currents are also present simultaneously on the line if it is carrying load, in which case the results we obtain may be superposed upon them. For simplicity we shall consider the line of zero leakage, this being accurate for aerial power transmission lines below corona potential. The problem of the damping of surges by corona involves variable parameters and is another story.[1]

Consider the system of Fig. 68, in which we have inserted at a a fictitious battery E in series with the line. This will produce, when transients have subsided, a steady potential distribution as shown in the diagram. Let us now insert suddenly at a an electromotive force E directed in the opposite direction. The two applied electromotive forces cancel; and we have thus artificially produced conditions duplicating those of the actual problem, for we have a charge initially localized on a part of the line and then suddenly released and allowed to spread. To solve we need simply to compute the potential waves produced by this suddenly-applied e.m.f. and superpose them upon the steady distribution of potential of the diagram. We can also compute the current waves, and as there is zero initial current, the result will be the actual net current distribution.

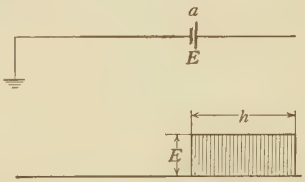

FIG. 68. Transmission Line with Initially Bound Charge.

The problem is now reduced to that of the sudden insertion of E between two parts of a transmission line. To get

[1] M. F. Gardner, Corona Investigation on an Artificial Line. Trans. A.I.E.E., **44**, 1925, p. 897.

effects during the first brief interval, in fact up to the time of the first reflection, it is accurate to consider the parts of infinite extent. The system of Fig. 69 is then analyzed.

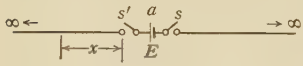

FIG. 69

Since ground impedance is neglected and since the circuit is symmetrical, it can also be arranged as in Fig. 70. The potential e for this figure now represents the sum of the potentials to ground at x and $-x$ of Fig. 69. These two are equal by symmetry, and a region halfway between lines of Fig. 70 can be considered to be at ground potential. This can be visualized by considering that the mid-point of the battery of Fig. 70 is grounded, and the switches S and S' are closed simultaneously. Thus,

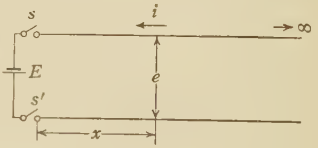

FIG. 70

$$e = E\epsilon^{-\beta x}\, 1. \tag{642}$$

In this

$$\beta = \sqrt{(2r + 2lp)\frac{c}{2}p} = \sqrt{(r + lp)\,cp}. \tag{643}$$

The resistance and inductance are doubled and the capacitance halved because we consider two lines in place of a line with ground return. Consequently β remains as in previous cases. In the current expression, however,

$$i = \frac{E}{2}\epsilon^{-\beta x}\sqrt{\frac{cp}{r + lp}}\, 1 \tag{644}$$

the factor 2 appears as a result of this modification. These expressions give the voltage and current waves for the configuration of Fig. 70. Those for the arrangement of Fig. 69, with which we actually deal, are identical, only half of the voltage appearing between either half of the line and ground.

Thus a wave of voltage $\frac{E}{2}\epsilon^{-\beta x}\,1$ runs from a toward the left, and the negative of this wave runs toward the right, these waves of voltage being superposed upon the distribution of voltage initially present.

These operational expressions have already been solved. For present purposes it will be sufficient to consider for a moment the simple case where $r = 0$. Then

$$\left.\begin{aligned} e &= \frac{E}{2}\epsilon^{-\sqrt{lc}\,xp}\,1 = \frac{E}{2}\epsilon^{-\frac{xp}{v}}\,1 \\ i &= \frac{E}{2}\sqrt{\frac{c}{l}}\,\epsilon^{-\frac{xp}{v}}\,1. \end{aligned}\right\} \tag{645}$$

The operators are now simply transfer operators and therefore rectangular waves of voltage and current travel at

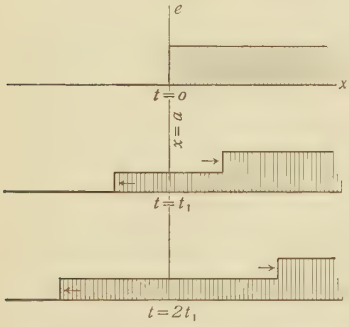

FIG. 71. The Traveling Waves when a Bound Charge is Released.

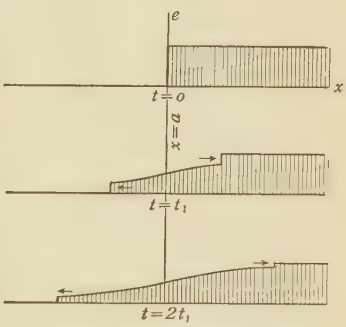

FIG. 72. Exaggerated Distortion of the Traveling Waves.

velocity $v = \frac{1}{\sqrt{lc}}$ along the line. Superposing the voltage wave on Fig. 68, there will result the successive diagrams of voltage distribution for increasing intervals of elapsed time shown in Fig. 71. If the complete expression for the voltage wave is used in place of the approximate expression obtained

from $r = 0$ above, the waves are modified in shape, and the wave fronts decrease in height by a factor $\epsilon^{-\rho t}$. The voltage diagram then appears as in Fig. 72, where the distortion is exaggerated in order to show it clearly. The current diagram is shown in Fig. 73.

When the bound charge is initially in a local region remote from the ends of the line, the phenomenon can be analyzed by treating both ends of the region as above, and the diagram of Fig. 74 will result. The initial voltage distribution divides

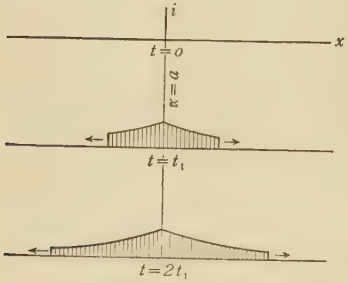

FIG. 73. The Traveling Current Waves.

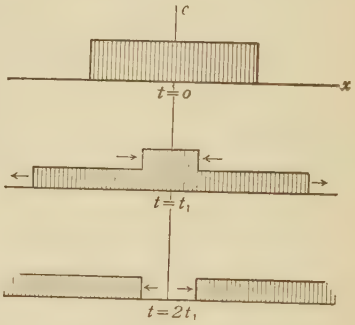

FIG. 74. Release of a Bound Charge in the Middle of a Line.

into parts which travel off to left and right. The terminals of the line may now be taken into account, and the reflection of the waves at these terminals examined in the usual manner. Actually, of course, the potential due to the induced charge on the line is not uniform over a region, nor abruptly terminated at the edges. The waves that start over the line are therefore not of perpendicular wave front. Moreover, the initial shape of lightning surges is not accurately known. When it is desirable to treat the propagation of waves initially non-perpendicular, the methods treated below may be employed.

4. **An Inductance Initially Carrying Current.** When the network contains an inductance initially carrying current,

an analogous expedient may be employed to place the problem in terms of a suddenly-applied electromotive force. A fictitious battery is first introduced into the network, if necessary, making it of such e.m.f. and so placing it that it will steadily maintain the system of currents initially present. A second e.m.f. equal and opposite to the first is then suddenly introduced. The resulting response of the network is exactly equivalent to the subsidence that would have ensued due to the energy initially present in the inductance. This is true because when both batteries are in the circuit their effects annul each other exactly and the process of decay of such currents as existed initially proceeds exactly as though they were not present. Of course, if any external voltages are applied at the instant of operating the switch which starts the transient, their individual effects are also computed and superposed.

A simple example will illustrate the proposition as well as a more complicated one. In Fig. 75 is shown a relay arrangement used in an automatic control. When the current in Relay *2* is I_0, Relay *1* operates and closes its contacts. The current in Relay *2* now decreases, and when it reaches I_1 this relay also operates and closes its contacts. We are interested in the current which passes through the winding of Relay *3*.

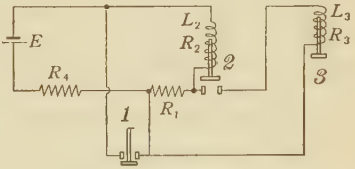

FIG. 75. A Relay Problem.

Evidently the battery and resistance R_4 are out of the network entirely during the time we are interested in, contact resistances being considered negligible. Also it is evident that the current in *2* decreases exponentially from I_0 to I_1, and that we need to consider conditions only after contacts *2* close. The simplified network is then shown in Fig. 76. In this we have inserted fictitious batteries E_1 and E_2, supplying electromotive forces I_1R_1 and I_1R_2 respectively. Together

it is evident that they produce steadily the exact system of currents in the network as existed at the instant contacts *2* closed.

It is now necessary to compute the effect of the sudden insertion of equal and opposite batteries at these points, as shown in Fig. 77, and combine the results with the initial

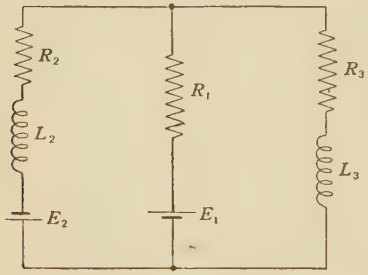

FIG. 76

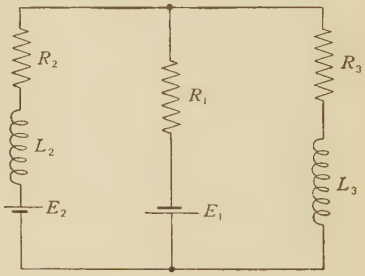

FIG. 77

currents. In this problem the current in the branch we are interested in is zero. The current component in Relay *3* due to E_1 is formulated by combining impedances in series and parallel:

$$i_1 = \frac{E_1(R_2+L_2p)}{(R_1R_2+R_1R_3+R_2R_3)+(R_1L_3+R_2L_3+R_1L_2+R_3L_2)\,p+L_2L_3p^2}1 \tag{646}$$

and that due to E_2 is

$$i_2 = -\frac{E_2R_1}{(R_1R_2+R_1R_3+R_2R_3)+(R_1L_3+R_2L_3+R_1L_2+R_3L_2)\,p+L_2L_3p^2}1. \tag{647}$$

Adding these algebraically and inserting the values of E_1 and E_2:

$$i = I_1\frac{R_1L_2p}{(R_1R_2+R_1R_3+R_2R_3)+(R_1L_3+R_2L_3+R_1L_2+R_3L_2)\,p+L_2L_3p^2}1. \tag{648}$$

This is familiar, and

$$i = \frac{I_1R_1}{L_3\omega}\epsilon^{-\alpha t}\sinh\omega t \tag{649}$$

where

$$\alpha = \frac{1}{2L_2L_3}(R_1L_3+R_2L_3+R_1L_2+R_3L_2)$$

$$\omega = \sqrt{\frac{1}{4L_2^2L_3^2}(R_1L_3+R_2L_3+R_1L_2+R_3L_2)^2 - \frac{1}{L_2L_3}(R_1R_2+R_1R_3+R_2R_3)}.$$

This problem has been carried through in detail, for it illustrates the advantage of keeping expressions in operational form until all combinations are made. Here it avoids working out four exponentials and adding point by point.

5. **A Sudden Short Circuit in a Network.** When the transient in a network is caused by the sudden short circuiting of a portion of the network, the problem can be restated in terms of the sudden application of a unit e.m.f. The network is assumed to be initially in the steady state with various steady currents flowing and various steady voltages present. Between any two points of the network PP',

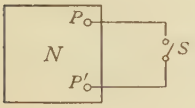

FIG. 78

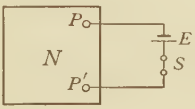

FIG. 79

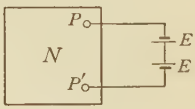

FIG. 80

Fig. 78, switch S is closed. Proceed as follows: Suppose the voltage E existed between P and P' before closing the switch. Introduce an opposite voltage E in the leads, as in Fig. 79, and close the switch. Evidently nothing will happen. A second voltage E opposing the first (Fig. 80) is suddenly introduced and the effect of this suddenly-impressed voltage computed. The result is the transient we seek, for the voltage PP' has suddenly been reduced from E to zero, and that is exactly what happens when the short-circuiting switch of Fig. 78 is closed. The currents and voltages computed are to be added to those initially present, to find the total result.

To illustrate, consider the transmission line of Fig. 81 which is open at the distant end and has electromotive force E applied at the home end, and assume that a short circuit is suddenly applied at the distant end. Leakage is neglected, so the voltage present across the switch before it is closed is E. First the system is reduced to that of Fig. 82, which has the same steady condition as that of Fig. 81. Now examine the transient ensuing upon introducing an opposing e.m.f. E at the distant end, Fig. 83. This problem has already been analyzed. Finally we add

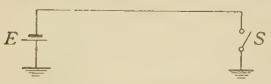

FIG. 81. Sudden Short Circuit of a Transmission Line.

FIG. 82

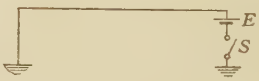

FIG. 83

the voltages found in Fig. 83 to those present in Fig. 82. The result is as shown in the successive diagrams of Fig. 84.

6. **Sudden Opening of a Switch.** The transient following the sudden opening of a switch in a network may be treated in a comparable manner. This problem can be reduced to terms of a suddenly-applied current. Suppose the switch of Fig. 78, initially closed, is to be suddenly opened. Let the switch carry an initial steady current I. We shall suddenly reduce this current to zero. This is equivalent to opening the switch. To effect this reduction to zero superpose upon the currents initially present in the network, those that would be produced by the sudden insertion at the position of the switch

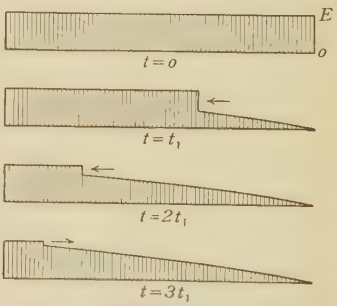

FIG. 84. The Traveling Waves of Voltage with Exaggerated Distortion.

of a constant-current generator supplying current I in the direction opposite to that current initially present, Fig. 85.

If the current to be interrupted flows through an inductance, and we thus propound the fictitious problem of what will happen when we suddenly impress a constant current on an inductive circuit we of course meet impulsive operators, for the voltage produced is mathematically infinite for an infinitesimal time. It is unnecessary to enlarge on this matter. As an illustration, consider the problem of the voltages produced on suddenly opening an inductive circuit. We consider that the switch actually interrupts the circuit abruptly. Of course it cannot do so practically unless there is a sufficient amount of distributed capacitance present to reduce the voltage which appears across the switch, as it separates, to a value below that which will produce and sup-

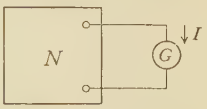

FIG. 85. A Network which is Suddenly Open Circuited.

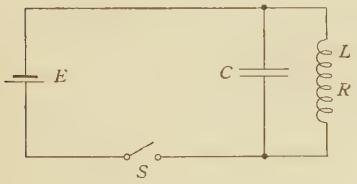

FIG. 86.

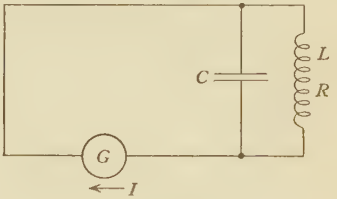

FIG. 87

port an arc. We shall thus avoid the impulsive problem by including the effect of the distributed capacitance. Consider the idealized case where the distributed capacitance is represented approximately by a lumped capacitance connected across the coil terminals, Fig. 86. For actual analysis we utilize the modified circuit of Fig. 87. The equation for the voltage across the parallel group is

$$e = \frac{I}{Cp + \dfrac{1}{R + Lp}} 1. \qquad (650)$$

Neglect resistance and this is

$$e = I\frac{Lp}{LCp^2+1}1 = I\sqrt{\frac{L}{C}}\sin\frac{t}{\sqrt{LC}}. \qquad (651)$$

The voltage which appears is sinusoidal, undamped when resistance is neglected, and of maximum value $I\sqrt{\frac{L}{C}}$. The effect of resistance in reducing this maximum is readily computed if desired.

The maximum voltage appears at a time $\frac{\pi}{2}\sqrt{LC}$ seconds after the contacts part. When an air-break switch opens, there is a race between the rate at which the contacts separate and the rate at which voltage builds up, the outcome of which decides whether there is or is not to be an arc. Something similar happens when an arc breaks.

If the switch at the distant end of the line of Fig. 81 is suddenly opened when carrying current, we may similarly compute the waves of voltage and current on the line. In this case the voltage which appears immediately at the switch is $I\sqrt{\frac{l}{c}}$. There is no time lag. The factor $\sqrt{\frac{l}{c}}$ for an aerial transmission line may be around 300 ohms. Several hundred amperes may be flowing. It is evident that an air-break switch will be likely to misbehave, and that an oil switch on an alternating-current line will draw an arc under oil although it should be extinguished when the current drops to zero. With high-voltage direct-current transmission lines some means will be necessary for opening short-circuit currents gradually or for protecting against undue voltage-rise by condensers or the like; for an abrupt opening on an unprotected line, caused for instance by a vacuum switch, can readily produce very large voltages indeed.

A great deal is yet to be written concerning the general problem of opening switches carrying current. If the opening is actually abrupt, it can be treated as indicated here.

The abrupt change (either an increase or a decrease) in the value of a circuit parameter can be handled by dividing the corresponding circuit element into parts and arranging a switch such that when it is opened or closed the value of the parameter active in the network is abruptly altered. The sudden insertion or removal of a complete circuit element may also be treated in this manner. Thus, if we are to suddenly insert an inductance at some point, we put the inductance in place with a short-circuiting switch across it, and then suddenly open the switch.

7. **Alternating-current Cases.** The necessity for analysis of the phenomena attendant upon the opening and closing of switches arises also when the initial steady state of the network is produced by sources of alternating electromotive force.

If we are to close a switch in such a network we follow the same procedure as above, except that the e.m.f. introduced suddenly is alternating, and is introduced at the point in the cycle which corresponds to the instant of switch closure. The effect may usually be computed best in two steps, finding first the response due to unit e.m.f. and then extending this result to that obtained from an alternating e.m.f. by employing the superposition theorem.

Similarly if a switch is to be opened, introduce a generator which supplies a constant alternating current, introducing it at that phase which corresponds to the instant of switch operation. Again treat in two parts the effect of a unit current and the extension to the effect of a constant suddenly-impressed sinusoidal current. We have not treated the use of the superposition theorem for suddenly-applied currents, but it works exactly as for suddenly-applied voltages.

Simple examples illustrate ideas without the confusion attendant upon those which are more complicated. Let us suddenly introduce, at $t = 0$, a current

$$i(t) = I \sin \omega t \, 1 \tag{652}$$

into a condenser through a resistance. For unit current,

$$e = \left(R + \frac{1}{Cp}\right) 1 = R + \frac{t}{C}. \tag{653}$$

Call this $V(t)$. Then for the actual sinusoidal current, by the superposition theorem:

$$\begin{aligned} e(t) &= i(0)\,V(t) + \int_0^t i'(\lambda)\,V(t-\lambda)\,d\lambda \\ &= \int_0^t I\omega \cos \omega\lambda \left(R + \frac{t-\lambda}{C}\right) d\lambda \\ &= IR \sin \omega t - \frac{I}{\omega c}(\cos \omega t - 1). \end{aligned} \tag{654}$$

Let us suddenly introduce the same current into an inductance, and neglect distributed capacitance. The voltage resulting from the introduction of unit current is then impulsive:

$$V(t) = (R + Lp)\ 1 = R + Lp\ 1. \tag{655}$$

From the complete form of the superposition theorem,

$$e(t) = i(0)\,\overline{V}(t) + \int_0^t i'(t)\,\overline{V}(t-\lambda)\,d\lambda + Li'(t) + Li(0)p\ 1. \tag{656}$$

In this expression currents replace voltages, and in the last two terms an inductance replaces a capacitance. We have also used $\overline{V}$ to signify the non-impulsive part of V, in conformity with the notation of Chapter V. Now $\overline{V} = R$, and $i(0) = 0$. So

$$\begin{aligned} e(t) &= \int_0^t RI\omega \cos \omega\lambda\, d\lambda + L\omega I \cos \omega t \\ &= RI \sin \omega t + L\omega I \cos \omega t. \end{aligned} \tag{657}$$

No infinite voltage appears when we introduce a sinusoidal current at the instant when the current is zero, even though the distributed capacitance is considered zero.

8. **Polyphase Networks.** Polyphase alternating-current

networks introduce no fundamentally new ideas, but some complication.[1] It will be unnecessary for us to delve into this complication in detail, for to do so would require a great deal of space and illustrate but little in the way of new operational processes. The treatment of polyphase networks is made in general by analyzing several individual transients by methods with which we are familiar, and combining these by superposition. Consider, for example, the case of a three-phase short circuit on a three-phase transmission line. If the network is as in Fig. 88 we are to treat the effect of suddenly closing simultaneously the three switches connecting the lines together. An equivalent effect is produced by suddenly introducing at these three points alternating electromotive forces of such phase and magnitude as to reduce to zero the voltages which existed at the moment of short circuit. This requires the computation of three individual transients, one for each switch, and the addition of all three of these to the initial steady state. Each transient is computed for a pair of two lines as a single-phase problem, and the actual total resultant current in any one line consists of the sum of the initial current it carried and the two transients computed for the pairs of which that line is a member. The process of computation of the individual transients depends of course upon the assumptions which are justifiable concerning the network parameters and connections. Thus if the lines are long we are concerned with a problem of distributed parameters and of traveling waves and terminal reflections. On the other hand if the lines are short, the parameters may be considered lumped.

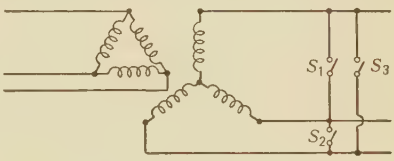

FIG. 88. A Polyphase Short Circuit.

[1] For a treatment of short-circuit calculations and steady-state theory of polyphase networks see O. G. C. Dahl, Electric Circuits — Theory and Application, McGraw-Hill, 1928.

In either case the parameters associated with the transformer bank are to be the parameters which control the impedance between a pair of terminals. These depend upon the nature of the transformer connections. Evidently the complication can be great indeed, but we do not need to treat here the matter of polyphase transformer connections and the resulting impedances.

In the steady-state analysis of a polyphase network which is balanced the computations can be made for a single line only, and the labor of computation is hence much reduced. This is possible because of the known fact that whatever happens in one phase or circuit will be exactly repeated in the next phase a third of a period later. Hence coupling terms in the equation for phase A, which would normally include expressions involving currents and voltages in phases B and C, can be written entirely in terms of effects in A alone. This reduction of a polyphase circuit problem to a single-phase problem cannot be carried through for transients, for effects in A are not exactly repeated in B and C. The network must therefore be computed as it stands.

In analyzing polyphase transmission-line transients we are also often concerned with the effect of the ground. Suppose a polyphase switch is to be closed connecting a grounded-neutral transformer bank to a transmission line. When the first switch point, say A closes, there is a transient between line A and ground. This is computed by the usual process of applying a unit e.m.f. and then employing the superposition theorem, using the inductance, resistance, and capacitance, as nearly as they can be estimated, of the circuit composed of line and ground return. When the next switch point closes there is a similar transient between B and ground, and also a transient between A and B involving the line-to-line parameters. When C closes there is a third line-to-ground transient, and two line-to-line transients, A to C and B to C. The sum of six terms thus gives the complete transient which results when the switch is closed.

It is evidently laborious to carry through the above computation, and machine processes of integration are practically

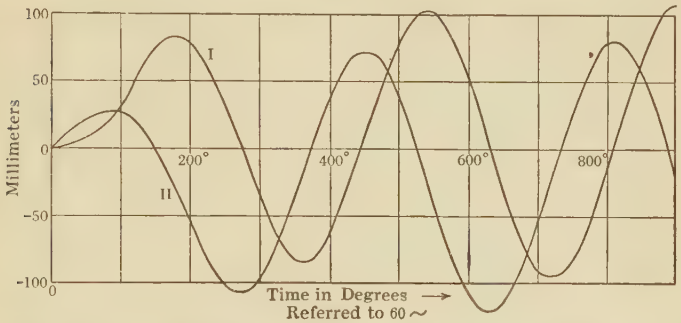

FIG. 89. The Component Curves Obtained in Calculating an Alternating-current Transient of a Polyphase Transmission Line.

essential. In Fig. 89 are shown two curves obtained by machine processes as intermediate steps in the calculation of the resultant curve shown in Fig. 90. The two curves

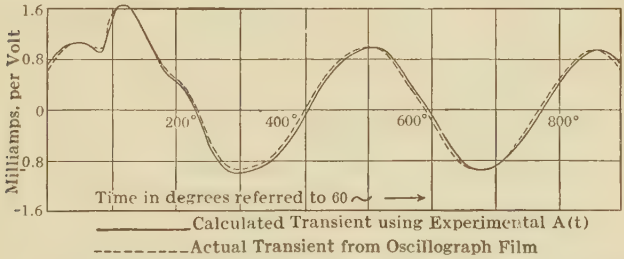

FIG. 90. Transient Current in One Phase of a Polyphase Artificial Transmission Line. The Three Line Switches were Closed Simultaneously.

represent the solution of the integrals $K\omega \int_0^t A(\lambda) \sin \omega\lambda \, d\lambda$ and $K\omega \int_0^t A(\lambda) \cos \omega\lambda \, d\lambda$, which are terms in the superposition theorem (105), Chapter V. The $A(t)$ used was

obtained experimentally on the artificial transmission line with far end grounded. These figures are taken from the Master's theses at the Massachusetts Institute of Technology of Messrs. Finley and Williams, 1926.

9. **Overlapping Transients.** We sometimes meet a problem in which one jolt is given to a network, and then before the network has recovered and subsided into a steady state a second jolt is administered. Thus we may, for example, apply an e.m.f. to a network and while the system is still in the transient state short circuit one of the circuit elements. There are evidently very many combinations of this sort that can be postulated. These problems differ from those we have considered, for every transient we have studied so far has represented the subsidence from an initial steady state to a final steady state. We now wish to break in on the progress of a transient and suddenly modify its conditions of existence in some way. Like the previous problems, these can be handled by reducing them to their equivalent problems in terms of suddenly-applied voltages, currents, or the like. It will be sufficient to illustrate the procedure for doing this by a simple case, for the same general idea will work for all.

Let us consider the network of Fig. 91, therefore, and suppose that we are to close switch S_1 at $t = 0$, and then at $t = t_1$ close S_2, short circuiting a portion of the network. We are interested in the current at some point of the network which we need not specify for our present purpose. First let us solve for the complete transient with S_2 left open, and then evaluate the voltage across this switch, designating it as $e(t)$. Consider next that we close both switches at the same time, but that we insert the e.m.f. $-e(t)$ in series with S_2 so that it prevents current flow through this connection. Evidently we shall get the previous solution.

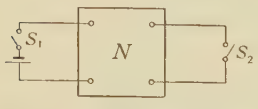

FIG. 91. Network Involving Overlapping Transients.

Finally, into this network, now closed everywhere, we shall consider the sudden introduction of the e.m.f. $e(t)$ at the time $t = t_1$; that is, we apply $e(t)\,1_{t_1}$. The result of this introduction, plus our first solution, gives the desired solution of the problem. This is evident, for we have produced the effect of no current through S_2 from $t = 0$ to $t = t_1$, and no voltage at this point thereafter.

As a very simple example of the above general type, consider the network of Fig. 92, where we shall solve for the battery current. The first solution gives

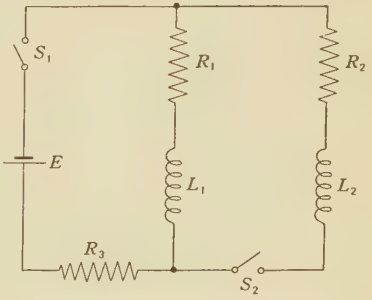

FIG. 92. Problem in which Second Transient is Started before First has Died Away.

$$i_1(t) = \frac{E}{R_1 + R_3}\left(1 - \epsilon^{-\frac{(R_1 + R_3)t}{L_1}}\right). \tag{658}$$

The voltage across the switch due to this transient is

$$e_2(t) = E\left(\frac{R_1}{R_1 + R_3} + \frac{R_3}{R_1 + R_3}\,\epsilon^{-\frac{(R_1 + R_3)t}{L_1}}\right). \tag{659}$$

To obtain the second solution we first find the transfer indicial admittance for the battery circuit when unit electromotive force is introduced at S_2. This is

$$\begin{aligned} A_{21}(t) &= \frac{1}{L_1 L_2}\,\frac{R_1 + L_1 p}{(R_2 + L_2 p)\,(R_1 + R_3 + L_1 p) + R_3(R_1 + L_1 p)}\,1 \\ &= \frac{1}{L_1 L_2}\left(\frac{M}{p + \alpha_1} + \frac{N}{p + \alpha_2}\right)1 \\ &= \frac{1}{L_1 L_2}\left[\frac{M}{\alpha_1}(1 - \epsilon^{-\alpha_1 t}) + \frac{N}{\alpha_2}(1 - \epsilon^{-\alpha_2 t})\right] \end{aligned} \tag{660}$$

where for abbreviation

$$\alpha_1, \alpha_2 = \left(\frac{R_2 + R_3}{2\,L_2} + \frac{R_1 + R_3}{2\,L_1}\right) \pm \sqrt{\left(\frac{R_2 + R_3}{2\,L_2} + \frac{R_1 + R_3}{2\,L_1}\right)^2 - \frac{1}{L_1 L_2}(R_1R_2 + R_2R_3 + R_1R_3)}$$

$$M = -\frac{R_1 - \alpha_1 L_1}{\alpha_1 - \alpha_2}, \qquad N = \frac{R_1 - \alpha_2 L_1}{\alpha_1 - \alpha_2}.$$

Now by the superposition theorem

$$i_{21}(t) = e(t_1)A(t - t_1) + \int_{t_1}^{t} A_{21}(t - \lambda)e_2'(\lambda)d\lambda \quad (661)$$

where the first term takes account of the abrupt change in $e_2(t)\,1_{t_1}$ at $t = t_1$, and the lower limit of integration is t_1 for the applied voltage is zero up to that time.
This gives

$$i_{21}(t) = \frac{E}{L_1L_2}\left[\frac{R_1}{R_1 + R_3} + \frac{R_3}{R_1 + R_3}\epsilon^{-\frac{(R_1 + R_3)t_1}{L_1}}\right] \times \left[\frac{M}{\alpha_1}(1 - \epsilon^{-\alpha_1(t - t_1)}) + \frac{N}{\alpha_2}(1 - \epsilon^{-\alpha_2(t - t_1)})\right] + \frac{ER_3}{L_1^2L_2}\int_{t_1}^{t}\left[\frac{M}{\alpha_1}(1 - \epsilon^{-\alpha_1(t - \lambda)}) + \frac{N}{\alpha_2}(1 - \epsilon^{-\alpha_2(t - \lambda)})\epsilon^{-\frac{(R_1 + R_3)\lambda}{L_1}}\right]d\lambda. \quad (662)$$

The integration of this gives, after reduction,

$$i_{21}(t) = \frac{E}{L_1^2L_2b}\left[R_1\left(\frac{M}{\alpha_1} + \frac{N}{\alpha_2}\right) - R_1\left(\frac{M}{\alpha_1}\epsilon^{-\alpha_1(t - t_1)} + \frac{N}{\alpha_2}\epsilon^{-\alpha_2(t - t_1)}\right) - R_3\epsilon^{-bt_1}\left(\frac{M}{\alpha_1 - b}\epsilon^{-\alpha_1(t - t_1)} + \frac{N}{\alpha_2 - b}\epsilon^{-\alpha_2(t - t_1)}\right) + R_3\left(\frac{M}{\alpha_1 - b} + \frac{N}{\alpha_2 - b}\right)\epsilon^{-bt}\right] \quad (663)$$

where $b = (R_1 + R_3)/L_1$.

Thus the battery current from $t = 0$ to $t = t_1$ is $i_1(t)$, and from $t = t_1$ on it is $[i_1(t) + i_{21}(t)]$.

This treatment illustrates an interesting use of the superposition theorem. When it has been used previously it has been assumed that the only abrupt change in $e(t)$ occurred at $t = 0$. In this problem there is an abrupt change at $t = t_1$ and the form of the expression is altered to take care of this. The extension to the case where there are several discontinuities in $e(t)$ can be readily carried out.

10. **Approximate Solution by Treatment of a Network in Parts.** Networks may often be split into parts for purposes of analysis. The simplest case in which this procedure is useful arises when we have a small circuit appended to a large one, small and large meaning carrying small and large currents. The current drawn by the small circuit may then produce a negligible effect on the performance of the large one and we may treat them one at a time, which is a great help. This needs only to be illustrated to show the process.

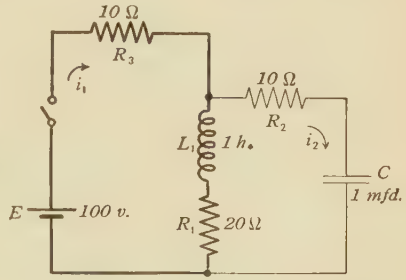

FIG. 93. Network Split into Parts for Analysis.

In Fig. 93 it is evident by inspection that the current i_2 which will pass in the circuit shown in light lines is small compared with the current i_1 in heavy lines. Thus,

$$i_1 = \frac{E}{R_{11}}\left(1 - \epsilon^{-\frac{R_{11}t}{L_1}}\right)$$

$$= \frac{100}{30}(1 - \epsilon^{-30t}). \qquad (664)$$

The voltage applied to the second circuit is the battery voltage minus the drop in the 10-ohm resistance, or

$$e_2 = 100 - \frac{1000}{30}(1 - \epsilon^{-30t}).$$

To obtain the current in the second circuit we can now use the superposition theorem:

$$i_2 = \frac{d}{dt}\int_0^t A(t-\lambda)\, e_2(\lambda)\, d\lambda \tag{665}$$

where $A(\lambda)$ is the indicial admittance of the second circuit alone, or

$$\frac{1}{10}\,\epsilon^{-10^5\lambda}.$$

So

$$i_2 = \frac{d}{dt}\int_0^t [100 - \frac{1000}{30}(1-\epsilon^{-30\lambda})]\,\frac{1}{10}\,\epsilon^{-10^5(t-\lambda)}\, d\lambda$$

which can be readily evaluated. Or we can apply the operator of the second circuit,

$$\frac{Cp}{R_2Cp+1}$$

directly to the voltage e_2:

$$\begin{aligned} i_2 &= \frac{Cp}{R_2Cp+1}\left(E - \frac{ER_1}{Lp+R_{11}}\right)1 \\ &= \left[\frac{ECp}{R_2Cp+1} - \frac{ER_1Cp}{R_2LCp^2+(R_{11}R_2C+L)p+R_{11}}\right]1. \end{aligned} \tag{666}$$

These are both recognizable operational expressions, so the result can be written out at once.

11. **Simplifying Assumptions Regarding Form of Applied E.M.F.** Sometimes it is amply accurate to replace one applied voltage by another which is simpler. This occurs in lumped circuits when the time constants of branches are widely different.

An interesting case of this comes up in the study of transmission-line surges in power systems. Suppose we have a line 100 miles long, and that the frequency of the system is 60 cycles per second. It takes a wave $\frac{1}{1\,8\,6\,0}$ second to run the length of this line. In a period of the applied voltage, $\frac{1}{6\,0}$ second, there is time for a wave to traverse the line thirty times. We are usually interested primarily in what happens

during the first, or at most the first few times that the wave runs back and forth on this line, say during $\frac{1}{600}$ second. In such an interval the applied voltage does not change greatly, particularly if it was applied at its peak value which usually furnishes the most interesting case. Hence it may be sufficiently accurate to treat the applied voltage as a constant, which greatly simplifies the mathematics.

In communication systems, on the other hand, circuits are sometimes so long in relation to the frequency involved that this procedure is unwarranted.

12. **Thévenin's Theorem.** There is a theorem, due to Thévenin,[1] which is often useful in setting up operational expressions. It was written for the steady-state performance of circuits, but it is just as useful operationally. It enables us to conveniently formulate the operational solution when one network J is joined onto another network K by connecting two points of the first with two points of the second, as in Fig. 94. The network J can be excited in any manner, as by closing a switch S. Suppose $e_p(t)$ is the voltage which appears at P with K disconnected and J excited alone. Then the theorem states that the current in the connection when K is added on and J is again excited is equal to $e_p(t)$ divided by the sum of the impedances looking into J and K respectively from the terminals at P. All of these factors may be expressed operationally.

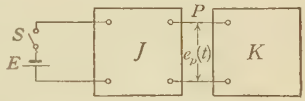

FIG. 94. Network Illustrating Use of Thévenin's Theorem.

Thus if $Z_j(p)$ and $Z_k(p)$ are the impedance operators for the networks examined from P, (S being closed and E absent in finding $Z_j(p)$, of course) and if $f(p)$ is the operator which applied to E gives the voltage at P when J is used alone,

$$i_p = E \frac{f(p)}{Z_j(p) + Z_k(p)} 1. \qquad (667)$$

[1] M. L. Thévenin, Sur un Nouveau Théorème d'Électricité Dynamique, Comptes Rendus, **97**, 1883, p. 159.

We are familiar with this theorem in studying machine behavior. There it states simply that the current is equal to the open-circuit voltage divided by the sum of the external and internal impedances. The expression is readily written when J is excited in some manner other than the above. Moreover the current at any point of K other than the terminal P may be evaluated by including the transfer operator which gives current at that point in terms of current at P.

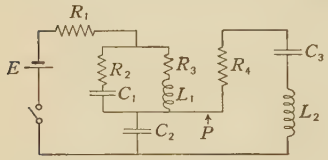

FIG. 95. Problem Analyzed by Thévenin's Theorem.

Suppose in Fig. 95 we wish the voltage across C_3 when the switch is closed. One way of setting up the expression is as follows: Consider the circuit open at P. The voltage which appears across C_2 is now

$$e_2(t) = f_1(p)E\,1$$

$$= \frac{1}{C_2 p}\left[\frac{E}{R_1 + \frac{1}{C_2 p} + \frac{\left(R_2 + \frac{1}{C_1 p}\right)(R_3 + L_1 p)}{\left(R_2 + \frac{1}{C_1 p}\right) + (R_3 + L_1 p)}}\right]1. \quad (668)$$

The impedances looking in from P are

$$Z_1(p) = \cfrac{1}{C_2 p + \cfrac{1}{R_1 + \cfrac{\left(R_2 + \frac{1}{C_1 p}\right)(R_3 + L_1 p)}{(R_3 + L_1 p) + \left(R_2 + \frac{1}{C_1 p}\right)}}} \quad (669)$$

and

$$Z_2(p) = R_4 + L_2 p + \frac{1}{C_3 p}.$$

The voltage at C_3 in terms of the current at P is given by

$$e_3(t) = \frac{1}{C_3 p}\, i_p\, 1. \tag{670}$$

Hence the operational equation which we seek is

$$e_3(t) = \frac{1}{C_3 p}\, \frac{f_1(p)}{Z_1(p) + Z_2(p)}\, 1. \tag{671}$$

This will of course simplify greatly on inserting the parts. Of course this particular result could have been obtained just as quickly without using Thévenin's theorem. The theorem is, however, occasionally useful.

13. **Reflection of Traveling Waves at Terminals.** This theorem can be used to much greater advantage in examining the reflection of a traveling wave at a terminal or junction of a transmission line. Consider first a very simple case such as that of Fig. 96 in which a network N is connected at the end of the line. An electromotive force E is suddenly applied at the home end. We shall examine the manner of reflection of the wave. For the present we shall simplify as much as possible while still illustrating the procedure. We shall neglect distortion, thus assuming that the wave arrives in rectangular form.

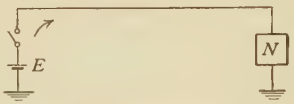

FIG. 96. Transmission Line Terminated by Network at Distant End.

Without the network at the distant end of the line, the voltage wave would be completely reflected and doubled when it arrived there, so a voltage 2 E would suddenly appear at the distant terminal. This gives the voltage to be utilized in applying the theorem. Since we are interested in the performance only during a brief interval after the wave arrives, the impedance of the line examined from the distant end may be formulated as for an infinite line. Hence, neglecting leakance,

$$Z_1(p) = \sqrt{\frac{r + lp}{cp}}. \tag{672}$$

Later, if we please, we can take account of reflections at the other terminal as we have done before.

If the network impedance is $Z_2(p)$ the current flowing from the end of the line through N will be, in accordance with the theorem,

$$i(t) = \frac{2E}{\sqrt{\frac{r+lp}{cp}} + Z_2(p)} 1 \tag{673}$$

and the distant-end voltage will be

$$e(t) = \frac{2EZ_2(p)}{\sqrt{\frac{r+lp}{cp}} + Z_2(p)} 1. \tag{674}$$

To be explicit, suppose the distant end of the line terminates in a transformer which is open circuited. Consider this transformer approximately as a pure inductance L, and neglect the resistance of the line so that the line impedance operator is simply $\sqrt{\frac{l}{c}}$. Then

$$e(t) = \frac{2ELp}{\sqrt{\frac{l}{c}} + Lp} 1. \tag{675}$$

This operator is recognized, and yields an exponential. Therefore

$$e(t) = 2E\epsilon^{-\sqrt{\frac{l}{c}}\frac{t}{L}}. \tag{676}$$

The terminal voltage thus jumps to $2E$ on arrival of the wave, and then dies away exponentially until the next wave arrives.

Since we have assumed a line without losses we can readily describe the way in which the voltage varies out along the

line. With an e.m.f., $e(t)$, applied to the end of such a line the entering current is $\sqrt{\frac{c}{l}}\,e(t)$, and the voltage at a distant point on the line is $\epsilon^{-\frac{xp}{v}}\,e(t)$; that is it duplicates the terminal voltage but is delayed by a time x/v. In the case being discussed this applied e.m.f. is the voltage $2E\epsilon^{-\sqrt{\frac{l}{c}}\frac{t}{L}}$ which occurs at the end of the line. We can thus construct the series of space diagrams of Fig. 97 which show the mechanism of a voltage reflection at a terminal inductance.

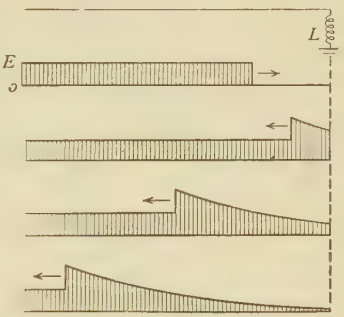

FIG. 97. Waves of Voltage on a Line Terminated by an Inductance.

The current at the end is

$$\frac{1}{Lp}e(t)\ 1 = 2E\sqrt{\frac{c}{l}}\left(1 - \epsilon^{-\sqrt{\frac{l}{c}}\frac{t}{L}}\right). \qquad (677)$$

Current diagrams are shown in Fig. 98.

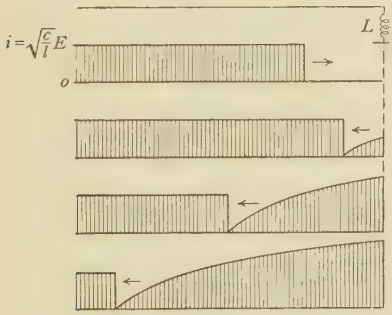

FIG. 98. Waves of Current on a Line Terminated by an Inductance.

14. **General Formula for Reflection and Refraction at a Junction or Terminal.** It will be worth while to derive the general relationships for the reflection and refraction of waves at a junction, load, or terminal of a line.

We shall pay attention to waves of voltage only, for a complete description of the progress of these is a complete treatment of our problem. The current waves if desired can always be readily obtained from them. We shall

examine a single-wire system with ground return. The two-wire system can be similarly handled.

Consider the system of Fig. 99. N is a network of lumped parameters connected at two terminals to transmission lines and at a third terminal to ground. N may be any sort of network, for example a transformer if its behavior is well enough represented by lumped parameters, or a filter for protecting terminal apparatus against surges, or a load on the line or the like. A particularly interesting example is the case where N represents a transformer connected between an aerial line and a power cable, for this combination in power networks has been a fruitful cause of breakdowns due to traveling waves coming in over the line caused by lightning or switching. Filters, which have been of great service in communication networks, are not as yet in real use for protecting against these troubles, largely on account of the lack of entirely satisfactory condensers, and also on account of the lack of any complete analysis of their proper design and possibilities.[1]

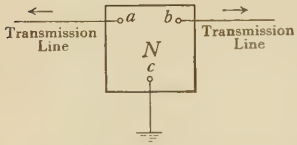

FIG. 99. A Network Inserted Into a Transmission Line.

We consider a wave of any wave form arriving from the left in Fig. 99, and we wish the entire analysis of what happens on each line after it arrives. The line on the right may be absent, if a terminal network is considered, in which case simply replace its impedance operator by ∞ in what follows.

[1] As a beginning in this direction see K. B. McEachron, Protection of Station Equipment on High-Voltage Transmission Lines, G. E. Review, **31**, 1928, p. 230; F. W. Peek, Jr., Lightning, Trans. A.I.E.E., **45**, 1926, p. 1131. Lightning and Other Transients on Transmission Lines, Trans. A.I.E.E., **43**, 1924, p. 1205.

Y. Satoh, Electric Oscillations in the Double-Circuit Three-Phase Transmission Line, Trans. A.I.E.E., **47**, 1928, p. 58.

An appreciation of the fundamental physical relationships involved in traveling waves is essential to a clear understanding of these matters. It is recommended that the reader consult Franklin and Terman, Transmission Line Theory, Franklin and Charles, 1926.

It is necessary to know the impedance operators of the lines and the network. Call the operator for the line to the left Z_1, writing

$$Z_1(p) = \sqrt{\frac{r_1 + l_1 p}{g_1 + c_1 p}} \tag{678}$$

and writing a similar expression for Z_2 for the line on the right. For the network we have the impedance operator which connects the voltage $a - c$ with the current entering a, which is formulated considering the second line connected at b. Thus

$$e_a = Z_a(p) i_a \, 1. \tag{679}$$

Also the operator which connects the voltages $a - c$ and $b - c$, is

$$e_b = F(p) \, e_a \, 1. \tag{680}$$

These operators are readily written when the connections and parameters of the network are known.

Now we shall consider that we have analyzed the simple problem wherein the left-hand line is open at a, and have found the manner in which the voltage at a will vary with the time when the incoming wave reflects from this open end. We shall count time from the instant the wave arrives, and write this open-circuit voltage as follows:

$$e_{a_1} = f(t) \, 1. \tag{681}$$

Using Thévenin's theorem, and considering the connections given in the figure, the current entering a after the wave arrives is then

$$i_a = \frac{f(t) \, 1}{Z_1(p) + Z_a(p)}. \tag{682}$$

The voltage at a is then

$$e_a = \frac{Z_a(p)}{Z_1(p) + Z_a(p)} f(t) \, 1 \tag{683}$$

and the voltage at b,

$$e_b = F(p)\frac{Z_a(p)}{Z_1(p) + Z_a(p)} f(t)\ 1. \tag{684}$$

If $\beta_1 = \sqrt{(r_1 + l_1 p)\ (g_1 + c_1 p)}$, and β_2 is similarly written for the right-hand line; and if x is the distance measured back along the left-hand line from a, and y is the distance measured along the right-hand line from b, we have then for the reflected wave of voltage:

$$e_x = \epsilon^{-\beta_1 x} e_a\ 1 = \epsilon^{-\beta_1 x}\frac{Z_a(p)}{Z_1(p) + Z_a(p)} f(t)\ 1 \tag{685}$$

and for the refracted wave of voltage

$$e_y = \epsilon^{-\beta_2 y} e_b\ 1 = \epsilon^{-\beta_2 y} F(p)\frac{Z_a(p)}{Z_1(p) + Z_a(p)} f(t)\ 1. \tag{686}$$

These are our complete solutions. The operators may of course be enormously complicated. Still we are usually interested only in small values of t, and we can obtain a few terms of the convergent series solution even in fairly complex cases.

There is hardly space to illustrate the above method by interesting examples; a chapter would be required to treat a single one, and no new operational processes would be involved.[1]

15. **The Bessel Cable. Heat Flow in Cylinders.** Heaviside treated a system which he called a "Bessel cable."[2] This is a cable in which the parameters c and $1/r$ are not constant along the cable but are proportional to the distance from one end. This is not a case of variable parameters for the parameters of the Bessel cable are fixed at any point in the cable, their values varying only with distance along the system.

[1] A large number of examples under simplifying assumptions are worked out by a slightly different method in the Mass. Inst. of Tech., Master's thesis of E. W. Boehne and L. H. Nothdurft soon to be published.

[2] Electromagnetic Theory, Vol. II. p. 211.

Lest one wonder what could possibly be of interest in a peculiar electrical system of this sort, it should be emphasized that we have been treating throughout not electrical circuits alone but circuits in general. We have used electrical terms for convenience, and because there are no terms which apply interchangeably to all sorts of circuits.

The principle of the Bessel cable is of extreme interest in connection with the problems of interior ballistics and of boilers and condensers. It has real application in problems of diffusion such as the radial flow of heat in cylinders.[1] Here the thermal capacitance and conductance per unit length vary directly with the radial distance from the axis of the cylinder. By thus using circular symmetry this problem of cylindrical flow is reduced to a one-dimensional problem. An analogous treatment, using parameters proportional to the square of a distance, could be used similarly to treat the diffusion in a sphere. The latter problem enters in physical chemistry in the solution of small crystals. Also there exist electrical systems with radial flow of current in dielectrics.

We treat therefore a cable in which

$$\frac{1}{r} = \frac{x}{r_0} \quad \text{and} \quad c = xc_0$$

where r_0 and c_0 are constants. The proportionality factor is adjusted by choosing the units in which x is measured. The positive direction of current will be chosen as directed toward $x = 0$. The differential equations now become

$$\frac{de}{dx} = \frac{r_0}{x} i, \qquad \frac{di}{dx} = c_0 x \frac{de}{dt} = c_0 x p e. \tag{687}$$

From these

$$\frac{d}{dx}\left(x \frac{de}{dx}\right) = r_0 c_0 x p e$$

[1] O. G. C. Dahl, Temperature and Stress Distribution in Hollow Cylinders, Trans. A.S.M.E., **46**, 1924, p. 161.

H. S. Carslaw, Heat Conduction, *loc. cit.*, p. 127.

or
$$\frac{d^2e}{dx^2} + \frac{1}{x}\frac{de}{dx} + \beta^2 e = 0 \tag{688}$$

where
$$\beta = j\sqrt{r_0 c_0 p}.$$

If we change variable by substituting $u = x\beta$ this becomes

$$\frac{d^2e}{du^2} + \frac{1}{u}\frac{de}{du} + e = 0 \tag{689}$$

which is a simple form of the Bessel differential equation.[1]

A solution of this is

$$e = AJ_0(u) + BK_0(u)$$

or
$$e = AJ_0(\beta x) + BK_0(\beta x) \tag{690}$$

where A and B are constants of integration, and J_0 and K_0 are Bessel functions of the zeroth order and first and second kind respectively. We shall treat the case of heat flow in a solid cylinder of radius a, with a varying temperature applied to the exterior surface. On account of the discontinuity in $K_0(\beta x)$ when $x = 0$ it follows that $B = 0$, and accordingly

$$e = AJ_0(\beta x). \tag{691}$$

To obtain the current, use

$$i = \frac{x}{r_0}\frac{de}{dx}$$

from which[2]
$$i = -A\frac{x\beta}{r_0}J_1(\beta x). \tag{692}$$

Dividing,
$$\frac{e}{i} = -\frac{r_0}{x\beta}\frac{J_0(\beta x)}{J_1(\beta x)}.$$

[1] See, for example, A. R. Forsyth, Treatise on Differential Equations, p. 159; D. A. Murray, Differential Equations, p. 105; or Byerly, Fourier Series and Spherical Harmonies, p. 13, and Chap. VII.
Gray, Mathews and Macrobert, Bessel Functions.

[2] Peirce, Table of Integrals, *loc. cit.*, p. 87.

This is the operational formula which connects the voltage and current at any point on the cable. The voltage is applied at $x = a$, and

$$i_a = -\frac{a\beta}{r_0}\frac{J_1(\beta a)}{J_0(\beta a)}e_a \tag{693}$$

gives the entering current. From

$$\left.\begin{aligned} e &= AJ_0(\beta x) \\ e_a &= AJ_0(\beta a) \end{aligned}\right\} \tag{694}$$

there results the operational formula for the voltage at any point in the cable,

$$e = \frac{J_0(\beta x)}{J_0(\beta a)}e_a. \tag{695}$$

In the heat problem this gives the temperature at any point in the interior of the material.

The treatment of these operators will serve to introduce some interesting operational processes which we have not as yet employed, and which we shall illustrate rather than attempt to treat completely, for they have not been rigorously justified. For convenience, write

$$\alpha_0 = \sqrt{r_0 c_0}$$

so that

$$J_0(\beta a) = J_0(j\alpha_0 p^{\frac{1}{2}} a). \tag{696}$$

The usual series are then

$$J_0(\beta a) = 1 + \frac{\alpha_0^2 a^2 p}{2^2} + \frac{\alpha_0^4 a^4 p^2}{2^2 \cdot 4^2} + \cdots \tag{697}$$

$$\beta a J_1(\beta a) = -\frac{\alpha_0^2 a^2 p}{2}\left(1 + \frac{\alpha_0^2 a^2 p}{2 \cdot 4} + \frac{\alpha_0^4 a^4 p^2}{2 \cdot 4^2 \cdot 6} + \frac{\alpha_0^6 a^6 p^3}{2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots\right) \tag{698}$$

and these are useful if the operand is continuous so that derivatives can be taken. For suddenly-impressed forces they are not immediately useful as they give rise, on division, to series in rising powers of p.

We can, however, use the asymptotic expression for the Bessel function.[1] When u is large,

$$J_n(u) = \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{(2n+1)\pi}{4}\right) \tag{699}$$

from which

$$\begin{aligned} i_a &= -\frac{a\beta}{r_0} \frac{\cos\left(\beta a - \frac{3\pi}{4}\right)}{\cos\left(\beta a - \frac{\pi}{4}\right)} e_a \\ &= \frac{a\beta}{r_0} \frac{\cos a\beta - \sin a\beta}{\cos a\beta + \sin a\beta} e_a \\ &= \frac{1 - \sin 2a\beta}{\cos 2a\beta} \frac{a\beta}{r_0} e_a. \end{aligned} \tag{700}$$

Now writing this in terms of p, it becomes

$$\begin{aligned} i_a &= \frac{1 - j \sinh (2a\alpha_0 p^{\frac{1}{2}})}{\cosh (2a\alpha_0 p^{\frac{1}{2}})} j \frac{a\alpha_0 p^{\frac{1}{2}}}{r_0} e_a \\ &= \left[\frac{a\alpha_0}{r_0} \tanh (2a\alpha_0 p^{\frac{1}{2}})\, p^{\frac{1}{2}} e_a + j \frac{a\alpha_0 p^{\frac{1}{2}}}{r_0 \cosh (2a\alpha_0 p^{\frac{1}{2}})} e_a\right]. \end{aligned} \tag{701}$$

It is legitimate to drop the last term, for i_a is real. Moreover, since we are using an asymptotic expression for the Bessel function, valid for large values of the argument, the tanh may be replaced by an expression which similarly represents it for large arguments, and this is simply unit. The tanh approaches unity exponentially, so there are no other terms in its asymptotic expression. This leaves

$$i_a = \frac{a\alpha_0}{r_0} p^{\frac{1}{2}} e_a. \tag{702}$$

If e_a is unit voltage,

$$i_a = a\sqrt{\frac{c_0}{\pi r_0}} t^{-\frac{1}{2}}\, 1. \tag{703}$$

[1] See, for example, J. Pierpont, Functions of a Complex Variable, Ginn & Co., 1914, Sec. 253.

Equation (703) is valid for small values of t only; that is, we have included only high-frequency components in our result. Notice that we have here dealt with the asymptotic expression for an operator, and not as previously with an operational expression which leads to an asymptotic expression for the result. In using an expression valid for p large we confine our attention to that range of the operator which takes into account only high-frequency components in the operand. Hence our result is good for small t, and gives no information concerning what happens when t is large. The initial form of current entering the Bessel cable is thus identical with that which will flow in an ordinary cable, the parameters being the same in each case at the home end.

We may treat our other operational expression similarly. Here, when $x \neq 0$, and for large values of β

$$e = \sqrt{\frac{a}{x}} \frac{\cos\left(\beta x - \frac{\pi}{4}\right)}{\cos\left(\beta a - \frac{\pi}{4}\right)} e_a = \sqrt{\frac{a}{x}} \frac{\cos \beta x + \sin \beta x}{\cos \beta a + \sin \beta a} e_a$$

$$= \sqrt{\frac{a}{x}} \frac{\cos \beta(x + a) + \sin \beta(x - a)}{\cos 2\,\beta a} e_a$$

$$= \sqrt{\frac{a}{x}} \frac{\cosh \alpha_0 p^{\frac{1}{2}}(x + a) + j \sinh \alpha_0 p^{\frac{1}{2}}(x - a)}{\cosh 2\,\alpha_0 a p^{\frac{1}{2}}} e_a. \qquad (704)$$

As before, drop the second term,

$$e = \sqrt{\frac{a}{x}} \frac{\cosh \alpha_0 p^{\frac{1}{2}}(x + a)}{\cosh \alpha_0 p^{\frac{1}{2}}(2\,a)} e_a \qquad (705)$$

and for large values of the argument,

$$e = \sqrt{\frac{a}{x}}\, \epsilon^{-\alpha_0 p^{\frac{1}{2}}(a - x)} e_a. \qquad (706)$$

We derived, (592)

$$\epsilon^{-\gamma^{\frac{1}{2}} p^{\frac{1}{2}}}\, 1 = 1 - erf \sqrt{\frac{\gamma}{4\,t}}. \qquad (707)$$

For unit voltage applied at the home end, therefore,

$$e = \sqrt{\frac{a}{x}}\left(1 - erf\frac{a-x}{2}\sqrt{\frac{r_0c_0}{t}}\right) \tag{708}$$

when t is small, and $x \neq 0$.

Comparing this with the result for an ordinary cable, we note that the factor $\sqrt{\frac{a}{x}}$ takes account of the curvature in radial diffusion. Otherwise we have the same sort of expression for the voltage distribution when t is small.

Now returning to the convergent expansions of the operators, we can obtain approximate expressions good for large t. The final values which are approached are obtained by inserting $p = 0$. This gives

$$i_a = 0, \quad e = e_a$$

so the current approaches zero, and the potential everywhere approaches that at the home end.

If terms in p^2 and those with higher exponents are dropped but those in p retained we have

$$i_a = \frac{2\,\alpha_0^2a^2p}{r_0(4 + \alpha_0^2a^2p)}1 = \frac{2}{r_0}\frac{p}{p + \frac{4}{\alpha_0^2a^2}}1$$

$$= \frac{2}{r_0}\epsilon^{-\frac{4t}{\alpha_0^2a^2}}. \tag{709}$$

Thus the current approaches zero approximately in accordance with this exponential, which shows why the operator could not be treated in the usual manner to obtain a result in the form of an asymptotic series.

A closer approximation can be obtained by retaining the p^2 terms. It happens that this gives a simple expression:

$$i_a = \frac{4}{r_0}\left[\frac{p^2 + \frac{8\,p}{\alpha_0^2a^2}}{p^2 + \frac{16\,p}{\alpha_0^2a^2} + \frac{64}{\alpha_0^4a^4}}\right]1$$

$$= \frac{4}{r_0} \frac{p^2 + \frac{8p}{\alpha_0^2 a^2}}{\left(p + \frac{8}{\alpha_0^2 a^2}\right)^2} 1$$

$$= \frac{4}{r_0} \frac{p}{p + \frac{8}{\alpha_0^2 a^2}} 1 = \frac{4}{r_0} \epsilon^{-\frac{8t}{\alpha_0^2 a^2}} \tag{710}$$

which is good for somewhat smaller values of t. We can thus proceed, with much complication it is true, to obtain as good an approximation as necessary.

In like manner for the other operator, dropping terms in p^2 and those with higher exponents,

$$e = \frac{1 + \frac{\alpha_0^2 x^2}{4} p}{1 + \frac{\alpha_0^2 a^2}{4} p} 1 = 1 - \left(1 - \frac{x^2}{a^2}\right) \epsilon^{-\frac{4t}{\alpha_0^2 a^2}}. \tag{711}$$

Retaining p^2, we can readily obtain

$$e = \frac{x^4}{a^4} \frac{\left(p + \frac{8}{\alpha_0^2 x^2}\right)^2}{\left(p + \frac{8}{\alpha_0^2 a^2}\right)^2} 1 \tag{712}$$

which after a few transformations becomes

$$e = 1 - \left(1 - \frac{x^4}{a^4}\right) \epsilon^{-\frac{8t}{\alpha_0^2 a^2}} - \frac{8}{\alpha_0^2 a^2} \left(1 - \frac{x^2}{a^2}\right)^2 t \epsilon^{-\frac{8t}{\alpha_0^2 a^2}}. \tag{713}$$

We thus have derived alternative expressions good for computation respectively when t is small or large. As usual there is a range of t over which neither expression is convenient, but if necessary the last process can be carried through to cover this region.

Solutions of this problem can be obtained by other processes, for example by treating the operators by contour integral methods. This section is intended rather to illus-

trate an operational procedure which seems to have considerable promise although as yet no definite formal justification.

PROBLEMS ON CHAPTER XIV

Prob. 1–14. A rectangular wave running down a single phase of a transmission line, considered to be without dissipation, encounters a condenser connected between line wires. What is the shape of the wave which passes beyond the condenser?

Prob. 2–14. A steady e.m.f. of value E is suddenly impressed on a network at time $t = t_1$, and suddenly removed at $t = t_2$, the point where the e.m.f. is applied being short circuited except when the e.m.f. is present. Write the general operational expressions for the response. Illustrate on the simple inductive circuit.

Repeat this analysis but with the network open except when the e.m.f. is present.

Prob. 3–14. A rectifier, as a circuit element, may be considered, providing minor effects are ignored, as a device which alternately short circuits and opens the network at the point where it is located, acting each time the voltage across it or the current through it drops to zero. Outline an operational attack on a network containing a rectifier. Illustrate by means of a circuit containing an alternating e.m.f., an inductance, a resistance, and an idealized rectifier.

Prob. 4–14. What general conditions may be stated regarding the form of the impedance function of a lumped parameter network if it is to be possible to insert a steady e.m.f. at one point in the network, and then subsequently open the circuit suddenly at this point without producing voltages that are mathematically infinite?

Prob. 5–14. A power cable is buried directly in the ground, and the whole is at uniform temperature. The cable is now placed in service and there is a loss of w watts per foot of length. After a lapse of h hours the cable is again taken out of service. The thermal conductivity of the cable material can be properly considered large compared with that of the surrounding earth. Outline an operational attack on the problem of the flow of heat about the cable.

Prob. 6–14. In Prob. 5–14 re-examine the conditions if the cable lies in a duct so that there is a thermal resistance, assumed constant, between it and the earth.

Prob. 7–14. An automatic household refrigerator, of known wall area, thickness, conductivity, and thermal capacitance, and with a known total thermal capacitance of contents which are assumed to be

always at uniform temperature throughout the interior, is operating in the steady state with an external temperature t_1 and an internal temperature t_2. The refrigerating unit suddenly stops operating. How will the internal temperature vary, and how would you evaluate the time to reach a given temperature t_3? Give the electrical analogue.

Prob. 8–14. If the refrigerator of Prob. 7–14 is operating steadily and the door is then opened for k minutes, how can the internal temperature variation be examined, and what is the electrical analogue? For a first approximation it can be assumed that the heat transfer from the interior through the open door is proportional to the temperature difference. What would be a more reasonable approximation?

Prob. 9–14. In Prob. 16–2 and Prob. 10–11 was considered the dissolving of a crystal. What is the form of treatment if the crystal is in the form of a long cylinder, the diameter of which can be considered to remain constant over the period of time considered? Suggest a scheme of attack for the case where the variation in diameter is actually taken into account.

Prob. 10–14. A penstock, carrying water to a power plant, is delivering a steady stream, when suddenly a butterfly valve at the delivery end slams shut. What maximum pressure will be produced, and approximately how long will it be maintained if the pipe is not ruptured? Specify exactly the parameters that are involved, and the electric-circuit analogue.

Prob. 11–14. In the previous problem suppose there is a vertical pipe connected into the main pipe just before the valve as a protection against water hammer. The water in this vertical pipe is stationary before the valve closes, and stands at a height sufficient to balance the head in the main pipe. The top of the auxiliary pipe is open. What is now the electrical analogue? What will be the effect of this auxiliary pipe on the maximum instantaneous pressure produced? What will be the approximate effect on the duration of the high pressure?

Prob. 12–14. In the previous problem consider that the top of the auxiliary pipe is closed, and that air pressure is forced in until the water is depressed nearly to the main pipe, a safety valve for the air being supplied so that the air pressure can rise but little beyond its steady value. Will this help in reducing the maximum instantaneous pressure? What secondary phenomena would you expect to occur?

Prob. 13–14. Two large metal blocks are initially at different temperatures t_1 and t_2 throughout. They are then placed in close contact. Consider only heat flow perpendicular to the common face. What is the resulting variation of temperature in the blocks? They may be considered, for t small at least, to be infinite in thickness.

Prob. 14–14. A certain kind of fish A is living in equilibrium with a food supply, the maximum number of fish that can exist on this supply being K_1, and the rate of increase in the number of fish being proportional to the difference between the number existing at any time and K_1. At a certain time a new type of fish B is introduced, and these prey on fish A, so that they produce a rate of decrease of type A proportional to the number of fish of type B present. For simplicity we assume that the rate of increase of type B is proportional to the number of A present, and that the number of B initially introduced is negligibly small. What is the variation in the number of A fish after the introduction?

Prob. 15–14. A somewhat more difficult problem, but one more nearly corresponding to physical conditions, is given by the assumption that the rate of increase of B fish is proportional to the number of A in excess of a constant K_2, and that K_3 fish of type B are initially introduced.

Prob. 16–14. A company manufactures a small device in large numbers which it guarantees unconditionally for a year's service, and on January 1 it ships N of these. Of these a fraction N_1 return for replacement h days later, and the number returning for replacement falls off thereafter in accordance with an exponential function Ke^{-kt}. Replacements are made promptly, and the articles sent as replacement return again for replacement in accordance with the same law. Set up the operational expression for the variation in the numbers shipped as time goes on. During the year what is the ratio of number shipped to number sold? What will this ratio be if on January 1 a selling schedule is introduced and maintained of N units daily, and what will be the ratio after the business has reached a steady state? These results should be formulated operationally and an indication given of the scheme of computation which would be used in numerical cases.

Prob. 17–14. A new and superior automobile polish is perfected by a large company, which first tries its introduction with a definite continuous amount of local advertising in a restricted territory. It is found that when a quantity q is sold in this territory during a very brief interval there is a delay of a time t_1 and that then orders begin to come in as users make repeat orders and tell their friends, so that the demand builds up at first linearly and with a slope k. The company now begins general distribution with a proportionate amount of general advertising, and sells initially at the rate of Q per day. At what rate should it plan to build up its production if it assumes that the same thing will happen generally as locally, and assumes that during the first period of its plans saturation of the market can be neglected? Give the operational expressions, and describe the curve of demand.

Prob. 18–14. A certain chemical reaction proceeds in two stages, material A decomposing to form B, and B decomposing in turn to form C. The rate of decomposition of A is proportional to the total amount of A present, and the same relation also holds for B. (See, for example, Mellor, Chemical Statics and Dynamics.) A solution is initially pure A, and it is required to obtain the manner in which C is formed. Set up an analogous electric circuit, and formulate the solution. Some excellent problems of this sort will be found in Phillips, Differential Equations, or Hitchcock, Chemical Differential Equations.

CHAPTER XV

NETWORKS WITH VARIABLE PARAMETERS

We have emphasized throughout that the networks with which we have dealt have been linear — that is, all parameters have been strictly constant. We shall now mention very briefly the treatment of networks in which one or more parameters are variable. The brevity is not due to any less importance of this branch of network analysis — on the contrary many of the most important and interesting problems of circuit theory involve variable parameters. The reason is that there is not much to present in regard to the operational treatment of such networks, for operational methods have not as yet been brought into practical use for anything but the linear case. If an attempt were made to treat the solution of differential equations with variable coefficients or the use of integral equations in problems of this nature, it might be possible to develop a mathematical treatment that would be both interesting and useful to the engineer and physicist. Operationally there is not much that can be presented along this line. Yet it will be worth while to mention the status of the problem of variable networks briefly, for in the operational solution of networks which we consider linear we are often using an approximation the legitimacy of which can be better estimated if we consider even superficially the effect produced when a parameter varies.

In Chapter I we have given a few examples of circuits in which the parameters vary. There are a host of practical problems of this sort, the simple solution of which would be an accomplishment of the first magnitude. To be useful, the solution must be so simple that it can be rapidly and accurately carried through in all sorts of cases; and this

is not at present possible. Any circuits utilizing thermionic tubes, rectifier circuits such as the mercury arc circuits now being introduced into this country for large power, the circuits of rotating machinery — all involve variable parameters. In some cases only steady-state solutions are needed, but often the transient solutions are required also if we are to effectively design and utilize circuits of this nature. Of course methods of attack are available, but they are in general indirect and cumbersome.

The present situation in regard to circuits with variable parameters may be compared with the status of alternating-current networks in the days before the practical development and use of vector diagrams and the symbolic method of solution. At that time methods were available for the solution of such networks in the steady state, for they had been developed by mathematicians in connection with the vibrations of mechanical systems. Even the analysis of distributed systems was available in connection with the exactly analogous problem of the vibration of strings. The theory of differential equations with constant coefficients, and even symbolic processes in connection with these, appeared in a large number of mathematical treatments. Yet the use of alternating currents in the hands of electrical engineers and physicists proceeded in a halting and ineffective manner until these mathematical processes had been interpreted and systematized into a sort of shorthand so that they could be readily applied in everyday usage. Imagine what would be our quandary if, when we wished to examine some simple alternating-current circuit in the steady state (say an induction motor), we had to write out the differential equations of the network and then by a direct examination of these predict performance such as the load-speed curve.

For the transient analysis of networks with fixed parameters we have simplified the differential equation or Fourier integral attack into an operational scheme which is a short-

hand method of tabulating and utilizing readily the results of a somewhat complicated method of analysis. Our operational methods thus occupy in this field much the position taken by the vector diagram in the field of steady-state analysis.

For the analysis of variable networks there exists at present no such convenient tool. The mathematical processes of analysis are available, but are yet to be interpreted and systematized for practical use. This is true not only for the transient analysis of such networks, but also even for their steady-state analysis. Unless the same thing can be accomplished in this realm that has been carried out in that of linear networks, we shall be circumscribed indeed in the attempt to utilize non-linear circuit elements in practical work.

2. **Summary of Method of Attack.** The principal schemes of attack on the variable network may be summarized as follows:

a. By various artifices by which such networks can be treated as though the parameters were fixed. Thus, for example, the equivalent sine wave and equivalent reactance schemes are used in the steady-state analysis of transformers containing iron, where the inductance is really highly variable. Another instance is the treatment of thermionic tube circuits, by assigning constants to the tube, such as the amplification constant and the mutual conductance.[1] These are really equivalent or average parameters, and analysis in terms of them is only valid when the range of current and voltage is small; when, so to speak, we remain on the linear portion of the tube characteristic. It is also common in thermionic tube circuit analysis to replace a variable resistance in the plate circuit by a fictitious alternating voltage in order to make analysis possible. This, in fact, is done so often that many undoubtedly do not realize that it is an approximation at all.

[1] Van der Bijl, Thermionic Vacuum Tube, McGraw-Hill, 1920.

Variable networks can sometimes be solved by the methods of linear networks by a change of variable. Proceeding in this manner, Bekku[1] and Ku[2] have reduced some of the problems of rotating machines to the problem of the static transformer, and have then handled an interesting transient study operationally. In these problems the mutual inductance between two circuits is almost a sinusoidal function of the time, and is assumed to be exactly sinusoidal for purposes of analysis. All other parameters are assumed fixed. If we attempt to solve for the currents directly, we have differential equations containing a current term multiplied by a variable coefficient which can be written in exponential form. If, however, we choose as a variable the product of the current and this exponential, then the equations become linear and we can proceed by the usual operational schemes. The shifting formula for transferring an exponential term is utilized to make this simplification possible.

In analyzing rotating machinery the principle of superposition is often used. The use of this principle when there

[1] S. Bekku, Researches of Electrotechnical Lab., Tokio, Japan, #203, June, 1927.

[2] Y. H. Ku, Transient Analysis of A.-C. Machines, to be presented at A.I.E.E. Midwinter Convention, 1929.

For other analyses of this problem, the reader is referred to

R. Rüdenberg, Kurzschlossströme beim Betrieb von Groszkraftwerken, Berlin, 1925; Elektrische Schaltvorgänge, Springer, Berlin, 1926.

R. E. Doherty, A Simplified Method of Analyzing Short-Circuit Problems, Trans. A.I.E.E., **42**, 1923, p. 841.

Doherty and Nickle, Synchronous Machines, Trans. A.I.E.E., **45**, 1926, p. 912; **46**, 1927, p. 1; **47**, 1928, p. 457.

Doherty and Shirley, Reactance of Synchronous Machines, Trans. A.I.E.E., **37**, 1918, p. 1209.

Doherty and Williamson, Short-Circuit Current of Induction Motors and Generators, Trans. A.I.E.E., **40**, 1921, p. 509.

V. Karapetoff, Initial and Sustained Short Circuits in Synchronous Machines, Trans. A.I.E.E., **44**, 1925, p. 403.

Further references can be obtained, if required, from the last paper which contains an excellent bibliography.

is a variable parameter present which is a function of the time is justified in a discussion by M. F. Gardner.[1]

b. By experimental analysis, or cut-and-try. This is the most common method of analysis when designing small-size units. Usually it is hardly dignified by the name analysis. Actually, however, it is not only a legitimate mode of procedure, but it is often the only sane way of going about a problem in the present state of the art. Suppose, for example, we have to design a small-size choke coil to carry a few milliamperes direct current and offer a given reactance to a small superposed alternating current. We can proceed some distance in analysis by the use of graphical methods involving the hysteresis loops of the iron employed. Yet the final design is usually carried out by building a few models of slightly different dimensions, testing and comparing these, and arguing in terms of general dimensional reasoning. It should be emphasized that from the engineering standpoint this may be the proper way of going about the matter. The question of costs enters into all engineering analysis as well as all engineering design. That method of analysis should be used on a given problem which will get the necessary results to the requisite degree of accuracy at least cost. It is sometimes well to use an elegant method for the sake of its elegance alone, but not in everyday engineering computation. Even an engineer puts many features in a structure for their æsthetic value rather than their utility, but the scaffolding is erected on the basis of utility and cost alone.

Thus if we wish to design a circuit for use with a particular thermionic tube as a detector in a radio receiver, we can often proceed to greatest advantage by gathering a collection of condensers and grid leaks and similar impedimenta and tinkering with the thing. The only difficulty with this procedure is that one who becomes adept at it is likely to

[1] Discussion at Midwinter Convention, 1929, A.I.E.E., of paper by Ku, *loc. cit.*

wish to adopt it as a cure-all, and try to design a 1000 kv-a. railway rectifier station in the same way.

The cut-and-try method is limited in utility to circuits which are cheap to build. Even then, a cut-and-try method guided and accompanied by all the analysis readily available is much more effective than one that goes blindly.

c. The classic treatment of differential equations with variable coefficients. This we can hardly go into.

d. The solution of the network by a scheme of successive approximations. This is powerful, and sometimes operational schemes are useful in making the successive steps. We shall treat it below briefly.

e. The use of integral equations.[1] The integral equation is a powerful and brilliant mathematical tool which is destined to be much better known than it is at present. When used on variable networks the difficulty comes in the solution of the equations rather than in their formulation. In fact the solution in a practical case usually drives us back to the use of successive approximations, which may be used directly almost as well as in connection with the integral equation. The integral equation formulation however is worth while, for it sums up the entire problem in a compact mathematical statement.

f. Machine methods of solving differential equations have recently had attention. Of course any machine that can solve differential equations with variable coefficients can be

[1] For the attack by integral equations see J. R. Carson, Theory and Calculation of Variable Electrical Systems, Phys. Rev., **17**, 1921, p. 116. Another discussion of variable parameters is given by H. W. Nichols, Theory of Variable Dynamical-Electrical Systems, Phys. Rev., **10**, 1917, p. 171.

The theory of integral equations is given in

V. Volterra, Leçons sur les Équations Intégrales, Gauthier-Villars, Paris, 1913; Rice Institute Pamphlet, Vol. 7, Oct., 1920.

M. Bôcher, Integral Equations, Cambridge University Tract No. 10.

W. V. Lovitt, Linear Integral Equations, McGraw-Hill, 1924.

A. G. Webster, Partial Differential Equations of Mathematical Physics, Teubner, 1927.

used for a study of variable networks. We shall not attempt to go into this subject here.[1] It is a fascinating subject, and one of considerable promise.

In Fig. 100 are shown solutions of the differential equations for the oscillations of a synchronous motor following a sudden change in shaft load. These solutions were obtained on the Integraph by Edgerton.[2]

g. Direct operational methods. It is easy enough to set up an operational expression for a variable network, but not much can be done with the expression after it is obtained. The operator will contain not only p, but also t in some manner. Weyl[3] has made some treatment of operators containing the time, but his methods are not yet in form

[1] See Bush, Gage and Stewart, A Continuous Integraph, J. Franklin Inst., 1927, p. 63; M.I.T., E.E. Research Bull. No. 56.

Bush and Hazen, Integraph Solution of Differential Equations, J. Franklin Inst., 1927, p. 575; M.I.T., E.E. Research Bull. No. 59.

Bush and Gould, Temperature Distribution along an Emitting Filament, Phys. Rev., 1927, p. 337.

K. E. Gould, Integration of a Functional Product by Measurement of Infra-red Radiation, M.I.T., Sc.D. Thesis, 1927. A New Machine for Integrating a Functional Product, J. Math. and Phys., **7**, 1929, pp. 305–316.

Finley and Williams, Analysis of Three-Phase Transmission Line Transients, M.I.T., S.M. Thesis, 1926.

G. S. Mikhalopov, Analysis of Transients on Three-Phase Transmission Lines, M.I.T., S.M. Thesis, 1926.

H. E. Edgerton, Abrupt Change of Load on a Synchronous Machine, M.I.T., S.M. Thesis, 1927.

L. Teplow, Stability of Synchronous Motors, M.I.T., S.M. Thesis, 1927; G. E. Review, **31**, 1928, p. 356.

F. J. Greuter, Transient Phenomena on a Three-Phase Artificial Cable, M.I.T., S.M. Thesis, 1927.

F. G. Kear, Integraph Solutions of Simultaneous Differential Equations, M.I.T., S.M. Thesis, 1928.

R. Cutts, Jr., Effect of Suddenly-Applied Load on the Shaft of a Synchronous Motor, M.I.T., S.M. Thesis, 1928.

C. E. Gewertz, The Pulling into Step of a Synchronous Induction Motor, M.I.T., S.M. Thesis, 1928.

N. F. Tsang, Surging of Synchronous Motor under Sudden Load, M.I.T., S.M. Thesis, 1928.

[2] *loc. cit.*

[3] H. Weyl, Gruppentheorie und Quantenmechanik, Hirzel, Leipzig, 1928.

for practical use. It may be that a tabulation of operational formulas can serve here to summarize solutions, but we have not yet advanced to that point. We shall illustrate the nature of these operators for variable networks, but cannot attempt any real treatment of them.

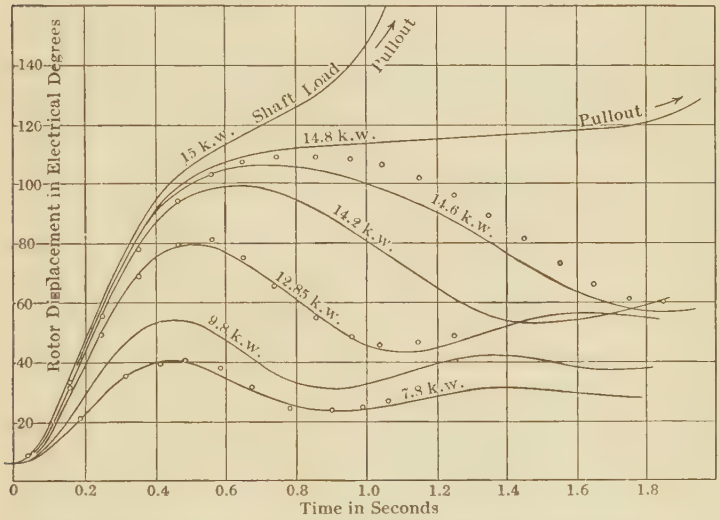

FIG. 100. Behavior of a Synchronous Motor When a Shaft Load is Suddenly Applied.

We need therefore only to illustrate the formulation in terms of operators or of integral equations and to outline roughly the scheme of successive approximations.

3. **Parameters Varying with the Time.** The simplest way in which a parameter can vary is as a function of the time only. An example, as mentioned above, is the alternator with mutual inductance which varies nearly as a sine function of t. Another example is the telephone transmitter with internal resistance which is arbitrarily varied by an external agency so that it can be considered a simple time function.

In the preceding chapter we considered a method by which it was possible to analyze, in terms of suddenly-applied voltages, the problem of the sudden change in the value of a network parameter. It would appear offhand as though, having analyzed for a unit abrupt change of this sort, we could get the effect of a gradual change by some sort of superposition theorem, just as we can get the effect of applying a gradually-varying voltage in terms of the effect of a unit voltage. So we can; but there is a complication here, and due to this complication we get an integral equation rather than simply a result in the form of an integral. This is due to the fact that as our parameter gradually changes, successive increments produce their effects in a network of altered properties instead of in a fixed network.

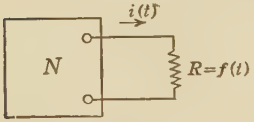

FIG. 101. Network with a Variable Parameter.

The simplest case of a varying parameter is that of a varying resistance. Consider the network of Fig. 101, which is assumed to contain sources of various sorts, and a resistance which varies and has the value $R = f(t)$. Let $A(t)$ be the indicial admittance of the remainder of the network viewed from this resistance. Call the current which flows through the resistance $i(t)$.

Then this current, flowing through the resistance, produces a voltage drop $i(t)f(t)$. This may be considered as an e.m.f. applied to the network, and in the superposition theorem we can introduce this expression in place of $e(t)$. The total current will be equal to the current which would flow if the variable parameter were short circuited, which we may call i_0, plus the current caused by the resistance variation. Thus

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t - \lambda) f(\lambda)\, i(\lambda)\, d\lambda \qquad (714)$$

is the integral equation of the problem. This is a Volterra type equation. In this everything is known except i,

which appears both on the left and under the sign of integration. The negative sign appears because the resistance drop tends to decrease the current. Other forms of the superposition theorem may be used, for example,

$$i(t) = i_0(t) - A(0)\, f(t)\, i(t) - \int_0^t A'(t-\lambda)\, f(\lambda)\, i(\lambda)\, d\lambda. \quad (715)$$

The same procedure can be readily carried out for the variation of some other parameter than a resistance. For a variable inductance $L = F(t)$ with a current $i(t)$ through it, the voltage drop is

$$L\frac{di}{dt} + i\frac{dL}{dt}$$

or

$$\frac{d}{dt}[F(t)\, i(t)] = F(t)\, i'(t) + i(t)\, F'(t) \quad (716)$$

and the integral equation becomes

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t-\lambda)\,[F(\lambda)\, i'(\lambda) + i(\lambda)F'(\lambda)]\, d\lambda. \quad (717)$$

If we wish to solve for the current at some other part in the network than the one where the variable parameter is located, we can first analyze as above; and, assuming we can find $i(t)$ from such an equation, we can compute from it the voltage across the variable element. Then the current in any part of the network can be found from the joint effect of this voltage and other sources, using of course the network without the variable element in this last computation.

This integral-equation formulation includes both the transient and the steady-state portions of the current. It is readily formed and very concise, but it is hard to evaluate.

An operational expression can be set up as well. Suppose $Z(p)$ is the impedance function of the balance of the network

as above, then for a resistance $R = f(t)$, an e.m.f. $-f(t)\,i(t)$ can be considered applied to the network, and

$$i(t) = i_0(t) - \frac{1}{Z(p)} f(t)\, i(t). \tag{718}$$

Now this can be written:

$$\left[1 + \frac{f(t)}{Z(p)}\right] i(t) = i_0(t) \tag{719}$$

where the expression $\frac{f(t)}{Z(p)}$ before an operand means that the latter is to be multiplied by $f(t)$ and then operated on by $\frac{1}{Z(p)}$.

We can even write

$$i(t) = \frac{i_0(t)}{1 + \frac{f(t)}{Z(p)}} \tag{720}$$

which gives an operator containing a time function, and very little in the way of rules to guide us. Of course the same information is involved in this expression as in the integral equation, so this is really only a shorthand way of writing the latter. In the operational expressions of this chapter the operand may be continuous if we are solving for steady-state performance, or it may be discontinuous if we are considering a transient. The latter case could be designated by appending the unit function to the operand, but this function can well be omitted here as both cases are handled similarly.

If the variable element be any combination of varying parameters the voltage across it can be written generally as $z(p,t)\, i(t)$. Thus for an inductance $L = F(t)$, $z(p,t) = F(t)p + F'(t)$. Thus,

$$i(t) = \frac{i_0(t)}{1 + \frac{z(p,t)}{Z(p)}}. \tag{721}$$

Such operators as this and the one above, however, no longer obey the algebraic laws.

Sometimes the variable parameter is the coupling parameter between circuits. This is the case, for example, in the rotating machine, where the mutual inductance between stator and rotor is a function of the time. Let subscripts s and r denote the stator and rotor circuits, and use $L_{sr} = f(t)$ as the variable mutual inductance. Then, by the same reasoning as above,

$$\left.\begin{aligned} i_r(t) &= i_{or}(t) - \frac{d}{dt}\int_0^t A_r(t-\lambda)\,\frac{d}{d\lambda}[f(\lambda)\,i_s(\lambda)]\,d\lambda \\ i_s(t) &= i_{os}(t) - \frac{d}{dt}\int_0^t A_s(t-\lambda)\,\frac{d}{d\lambda}[f(\lambda)\,i_r(\lambda)]\,d\lambda \end{aligned}\right\}. \qquad (722)$$

This is the single-phase case. For a polyphase case there are more equations, but they are readily written. These equations formulate the entire problem of alternators, induction motors, etc., both in the transient and steady states, with the single assumption that all parameters but the mutual inductance are fixed.

Since the principle of superposition holds for a network in which there is a variable parameter which is a time function we may find the effect of a complicated applied e.m.f. on such a network by splitting the e.m.f. into parts, solving for the effect of each part independently and then adding to obtain a correct total effect. There is little virtue in this process, however, for each partial solution must be obtained by solving an integral equation or its equivalent, so we may as well solve for the total effect at once. Superposition is actually useful when the joint effect of several independent sources applied at separated points in the network is desired. This case often arises in studying the transients of rotating machinery.

Although the principle of superposition holds for this type of network, the superposition theorem as we have used it does not. In a fixed parameter network the response

to unit e.m.f. is the same no matter at what instant the unit e.m.f. is applied, and this fact was used in deriving the superposition theorem. With the present type of network, due to the effect of the varying parameter, the response to unit e.m.f. depends upon the instant of application. While it is possible to derive a modified form of theorem taking this into account, there is no real utility in the process, for the necessity of solving an integral equation is not to be avoided in this manner.

To solve this type of network, therefore, when the applied e.m.f. is in series with a variable parameter which produces a voltage in accordance with

$$z(p,t)\, i(t) \tag{723}$$

it is necessary to find in some manner the solution of the integral equation

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t-\lambda)\, z(p,\lambda)\, i(\lambda)\, d\lambda. \tag{724}$$

4. **Parameters Depending on the Network Performance.** The variable parameter may be a function of the current through it or the voltage across it, or even of the current or voltage in some other part of the network. A transmission line operating above the critical voltage for corona formation has a leakage at any point which is a function of the voltage at that point, or more precisely of the entire past history of that voltage. The inductance of a coil with an iron core is similarly a function of the past history of the current. The resistance of a rectifier is a function — in fact, a discontinuous function — of the current through it. It can also be written as a function of the voltage across it. A perfect rectifier would be one in which the resistance would be zero for one direction of current and infinite when the current attempted to pass in the other direction. In a thermionic tube the current between the filament and plate is a function of both grid and plate voltages, although in a shielded-

grid tube the effect of the latter may be reduced to nearly zero.

If in Fig. 101 the resistance is a function of the current, $R = \phi[i(t)]$, the argument proceeds exactly as before, and the integral equation is

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t-\lambda)\,\phi[i(\lambda)]\,i(\lambda)\,d\lambda. \qquad (725)$$

This is a functional integral equation, for not only the unknown, but a function of the unknown appears under the integral sign.

If we wish to solve for the current at a distant point, and can solve this equation, we can proceed as in the case of a resistance varying with the time.

No matter what sort of variable element we put in place of the variable resistance above, if the voltage across it is a function of the current through it, say $e(t) = \psi[i(t)]$, we can write the integral equation of the problem:

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t-\lambda)\,\psi[i(\lambda)]\,d\lambda. \qquad (726)$$

For the variable resistance, $\psi(i) = i\phi(i)$. If the variable element is a rectifier, the function ψ (which is the rectifier characteristic) may be discontinuous. Figure 102 shows the form for a copper-copper oxide contact rectifier.[1]

When the value of the parameter depends upon the conditions at a remote point in the network, another complication arises. Suppose the voltage across an element located at M, that is e_M, depends on the voltage at N, so that $e_M(t) = F[e_N(t)]$. The voltage at N can be expressed in two parts, first that due to sources in the network, and second that due to the parameter variation:

$$e_N(t) = e_0(t) - \frac{d}{dt}\int_0^t H_{MN}(t-\lambda)\,F[e_N(\lambda)]\,d\lambda \qquad (727)$$

[1] Grondahl and Geiger, A New Electronic Rectifier, Trans. A.I.E.E., **46**, 1927, p. 357.

where H_{MN} is the solution for the voltage at N in the unvaried network due to a suddenly-impressed unit voltage at M. Having thus solved for the voltage at N, and hence for the voltage at M, we can find the currents throughout by the joint effect of the sources and the voltage e_M on the unvaried network.

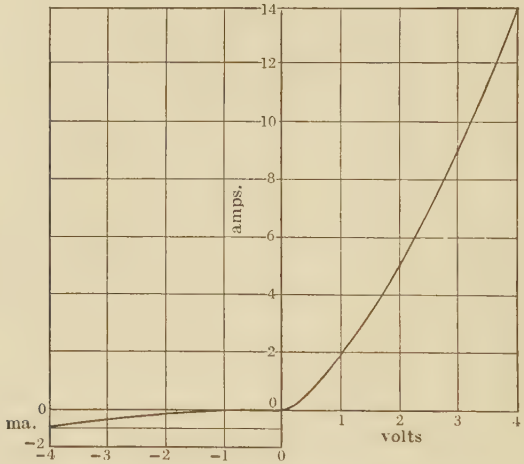

FIG. 102. Characteristic of a Copper–Copper Oxide Rectifier.

Thus we can readily write the integral equation of our problem even when it is rather complicated. It is an entirely different story when we try to solve it.

5. **Solutions by Successive Approximations.** A Volterra-type integral equation of the form:

$$i(t) = i_0(t) - \frac{d}{dt}\int_0^t A(t-\lambda)\, f(\lambda)\, i(\lambda)\, d\lambda \qquad (728)$$

has as a solution:[1]

$$i(t) = i_0(t) - i_1(t) + i_2(t) \ldots \qquad (729)$$

[1] Volterra, *loc. cit.*

where

$$\left.\begin{aligned} i_1(t) &= \frac{d}{dt}\int_0^t A(t-\lambda)\, f(\lambda)\, i_0(\lambda)\, d\lambda, \\ i_2(t) &= \frac{d}{dt}\int_0^t A(t-\lambda)\, f(\lambda)\, i_1(\lambda)\, d\lambda, \text{ etc.} \end{aligned}\right\} \tag{730}$$

This enables us to obtain a solution in the form of a series for any of the problems involving a parameter varying as a function of the time.

Let us examine in detail what we are doing here, paying attention (to be explicit) to the network with a varying resistance.

$i_0(t)$ is the current that will flow through the resistance branch if the variable resistance is absent. $f(t)\, i_0(t)$ is the voltage drop that will be produced if this first approximate current flows through the varying resistance. Then $i_1(t)$ is, from (730), the current that would be produced in the unvaried network by an e.m.f. of this form.

Similarly $f(t)\, i_1(t)$ is the voltage drop produced by this first correction term for the current, and $i_2(t)$ is the current that would be produced by the equivalent e.m.f. in the unvaried network, and so on.

Formally, therefore, the idea involved is quite simple. Unfortunately the calculations involved are likely to be rather complicated. They would be greatly simplified if we had available a machine that would directly evaluate and plot integrals of the form that occur in the superposition theorem.

We can carry out this same procedure operationally. Write the operational expression (720) in the form:

$$i(t) = \left(1 + \frac{f(t)}{Z(p)}\right)^{-1} i_0(t) \tag{731}$$

and expand by the binomial theorem:

$$i(t) = \left[1 - \frac{f(t)}{Z(p)} + \left(\frac{f(t)}{Z(p)}\right)^2 - \cdots\right] i_0(t). \tag{732}$$

Now operate on $i_0(t)$ term by term. The first term is $i_0(t)$. The second is

$$i_1(t) = \frac{1}{Z(p)}[f(t)\, i_0(t)] \tag{733}$$

and this is the current in the unvaried circuit which would be caused by an e.m.f. $f(t)\, i_0(t)$. The next term is

$$\begin{aligned} i_2(t) &= \frac{f(t)}{Z(p)}\,\frac{1}{Z(p)}[f(t)\, i_0(t)] \\ &= \frac{1}{Z(p)}[f(t)\, i_1(t)] \end{aligned} \tag{734}$$

and so on. This operational process is here justified simply because it leads to the same result as given by the rigorous integral equation treatment. It should not be concluded that any sort of operational procedure is valid that we happen to wish to employ. Various sorts of operations can readily be set up, and often results can be arrived at quickly in this manner, but they are usually only shorthand ways of carrying on, and when they are new they need justification.[1]

In the case where the resistance is a function of the current, we can proceed in a comparable fashion. Here we would write operationally:

$$i(t) = i_0(t) - \frac{1}{Z(p)}\psi[i(t)] \tag{735}$$

where $e(t) = \psi[i(t)]$ is the voltage-current characteristic of the variable resistance element. Write this

$$\left(1 + \frac{\psi}{Z(p)}\right) i(t) = i_0(t) \tag{736}$$

[1] For an example of the practical use of unusual operators, where the working rules are heuristically obtained and where interesting results are quickly arrived at, see W. O. Pennell, A General Operational Analysis, J. Math. and Phys., **7**, 1927, p. 24; M.I.T. Publication, Math. Dept. No. 137.

or, expanding as before,

$$i(t) = \left[1 - \frac{\psi}{Z(p)} + \left(\frac{\psi}{Z(p)}\right)^2 - \cdots\right] i_0(t). \qquad (737)$$

The first term is $i_0(t)$. The second is

$$i_1(t) = \frac{1}{Z(p)}\,\psi[i_0(t)] \qquad (738)$$

or the current that would be produced in the unvaried network by an e.m.f. $\psi[i_0(t)]$, that is the equivalent of the voltage set up when $i_0(t)$ flows in the variable circuit element. The next term will be similar, using $i_1(t)$ in place of $i_0(t)$, and so on. Evidently the physical reasoning is identical with that used in the preceding case, and the operational transformation is simply a way of summing up this physical reasoning. The result is not here justifiable by any simple comparison with the solution of the integral equation. In physical problems, however, it will usually be evident from physical considerations whether the series thus obtained will converge, and whether it is a reasonable solution to the problem in hand and hence of the integral equation which corresponds to it.

6. **Use of Operational Transformations in Complicated Cases.** We can use operational transformations to some extent in following even more complicated cases involving variable parameters. As a brief notation they are an aid for use in thinking out the physical reasoning, — on which latter we really depend to keep us straight. Thus for the rotating machine we can write in place of the integral equations (722):

$$\left.\begin{aligned} i_r(t) &= i_{or}(t) - \frac{pf(t)}{Z_r(p)}\, i_s(t) \\ i_s(t) &= i_{os}(t) - \frac{pf(t)}{Z_s(p)}\, i_r(t) \end{aligned}\right\} \qquad (739)$$

Let us confine our attention to a single-phase alternator in which there is a constant e.m.f. E applied to the rotor

or field circuit, and no voltage in the stator except that induced from the rotor. Then for steady-state analysis, $i_{or}(t)$ is E/R_r and $i_{os}(t) = 0$. The impedance $Z_s(p)$ includes that of the load circuit. The rotor current is

$$i_r(t) = \frac{\frac{E}{R_r}}{1 - \left(\frac{pf(t)}{Z_r(p)}\right)\left(\frac{pf(t)}{Z_s(p)}\right)} \tag{740}$$

and this can be written

$$i_r(t) = \left[1 + \left(\frac{pf(t)}{Z_r(p)}\right)\left(\frac{pf(t)}{Z_s(p)}\right) + \left\{\left(\frac{pf(t)}{Z_r(p)}\right)\left(\frac{pf(t)}{Z_s(p)}\right)\right\}^2 + \cdots\right]\frac{E}{R_r}. \tag{741}$$

The first term of rotor current is the constant value, E/R_r. The second is

$$\frac{pf(t)}{Z_r(p)}\left[\frac{1}{Z_s(p)}\, pf(t)\, \frac{E}{R_r}\right].$$

Now the quantity in brackets is the stator current that would be produced by the induced voltage if the field current remained strictly constant. The entire expression is the rotor current that would be induced if such a stator current flowed. Thus this second term is the rotor current due to a reflection into the stator and back. Similarly, the next term will give the component after two reflections, and so on.

If we are examining the transient behavior on closing the field switch, we have for $i_{or}(t)$ in place of E/R_r the transient expression $\frac{E}{Z_r(p)}\, 1$, which is the familiar exponential. The rest of the reasoning in terms of reflections proceeds exactly as before, and we get a succession of transient terms for the rotor current. Just the same procedure gives the stator current. Except for the great labor of evaluating expressions, we have here the entire transient analysis of a single-phase alternator.

Let us treat the steady-state case a bit further, however, and assume that the mutual inductance varies sinusoidally, for this will make clearer the physical ideas involved. The mutual inductance is, in general,

$$L_{sr}(t) = f(t) \tag{742}$$

so that the voltage in the stator due to a constant rotor current E/R_r is

$$\frac{E}{R_r} f'(t).$$

Now assuming the mutual inductance to vary sinusoidally, and using the symbolic scheme of notation,

$$L_{sr}(t) = L_{sr}\epsilon^{j\omega t}. \tag{743}$$

Then the induced voltage in the stator due to the first term of rotor current is

$$j\omega \frac{EL_{sr}}{R_r} \epsilon^{j\omega t}.$$

To this corresponds a stator current:

$$\frac{j\omega EL_{sr}}{R_r Z_s(j\omega)} \epsilon^{j\omega t}$$

which induces a rotor voltage:

$$\frac{d}{dt} \frac{j\omega EL^2_{sr}}{R_r Z_s(j\omega)} \epsilon^{2j\omega t} = -\frac{2\omega^2 EL^2_{sr}}{R_r Z_s(j\omega)} \epsilon^{2j\omega t} \tag{744}$$

and thus a second term of rotor current:

$$-\frac{2\omega^2 EL^2_{sr}}{R_r Z_r(j\omega) Z_s(j\omega)} \epsilon^{2j\omega t}$$

and so on.

The rotor current thus consists of a steady component plus a series of even harmonics. The expressions give these both in phase and magnitude. This, of course, is a well-known result for the rotating machine. It is interesting to

see that exactly the same reasoning will give the transient behavior if the enormous labor of carrying through in detail the evaluation of the expressions involved can be tolerated.

PROBLEMS ON CHAPTER XV

Prob. 1–15. Some types of radio detectors, notably the thermionic tube with certain connections, may be reasonably treated as variable resistances, the voltage across which varies as the square of the current. With such a detector in series with a source of alternating e.m.f. and a fixed impedance, harmonics are introduced, and this causes distortion. This problem may be analyzed for steady-state or transient behavior by integral-equation methods.

Prob. 2–15. The building up of voltage of a shunt or compound *d-c.* generator, when its field circuit is closed, is a problem in variable parameters on account of the saturation of the field winding. This problem is of much importance in connection with exciters supplying the field circuits of large alternators, and has received much attention. It has not been completely analyzed, taking all factors into account.

Prob. 3–15. A pendulum oscillating through a large angle involves a varying spring parameter. A slowly-turning machine with flywheel or other moving parts out of balance may be considered as a pendulum which swings all the way over. At higher speeds centrifugal forces are so important that gravitational effects may be ignored.

Prob. 4–15. Problems involving variable parameters are encountered frequently in connection with rectifiers, arcs, thermionic tubes, and glow discharges. One interesting problem is the formulation of the integral equations for a polyphase mercury-arc-rectifier network, where the arc characteristic in one phase depends upon simultaneous conditions in adjacent phases.

Prob. 5–15. A constant e.m.f. is suddenly applied to a circuit consisting of a unit inductance in series with a resistance of value proportional to the time after the switch is closed. Set up the integral equation. Solve by successive approximations, aided by operational formulation, and check the result by means of the differential equation of the circuit.

APPENDIX A

TABLE OF SYMBOLS

a = { The argument in the infinite integral theorem. / A constant.

a_h = Coefficient of Fourier series.

A = A constant.

$A(t)$ = Indicial admittance, the current resulting from unit e.m.f. suddenly impressed.

$A_{hk}(t)$ = Transfer indicial admittance, current in Circuit k when unit voltage is suddenly applied to Circuit h.

$\overline{A}(t)$ = Finite part of the indicial admittance, $A(t)$ less pulse.

b = { A decrement. / A constant.

b_h = Coefficient of Fourier series.

B = A constant.

$B(t)$ = $A(t) - P(o)$, the indicial admittance less the steady-state term.

c = { Capacitance per unit length of line. / A constant.

c_0 = A constant.

C = { Capacitance (in circuit with lumped parameters). / A path of integration.

C_0 = Capacitance of a terminal condenser.

$\overline{C}$ = Equivalent capacitance of a condenser group.

C_{hh} = Self-capacitance of the h Circuit of a network.

C_{hk} = Mutual capacitance of the h and k Circuits, current necessary in the k Circuit to produce unit rate of change of condenser voltage in the h Circuit.

D = { Length of a transmission line. / A constant.

$D(p) = \begin{vmatrix} z_{11}(p) & \cdots & z_{1n}(p) \\ \cdots & \cdots & \cdots \\ z_{n1}(p) & & z_{nn}(p) \end{vmatrix}$ Also the polynomial obtained upon simplifying the determinant.

$D'(\lambda) = \dfrac{d}{d\lambda} D(\lambda)$.

$\overline{D}(\lambda)$ = The determinant formed by placing all resistances equal to zero.

$e = e(t)$ = { A voltage between two points in a network. / Applied electromotive force. / Voltage on a transmission line at a distance x from generator end. } Functions of time.

e_C = Voltage across a condenser.

e_L = Voltage across an inductance.

e_0 = Voltage at generator end of a line.

E = { Voltage at $t = 0$. / Peak value of an alternating voltage. / Electromotive force of a source. } Not functions of time.

E_k = Peak value of an alternating voltage in Circuit k at $t = 0$.

E_0 = Voltage at generator end of line (a constant).

f = Frequency, cycles per second.

$f(x), F(x)$ = Functions of x.

$f'(x)$ = Derivative of a function.

g = { Conductance per unit length of line. / A constant (Chapter XII). }

G = Conductance (circuit with lumped parameters).

$G(\lambda) = A_{n-1}\lambda^{n-1} + A_{n-3}\lambda^{n-3} + \cdots + A_1\lambda$ (Chapter VII).

h = { Henries inductance. / Designation of any circuit of a network. / Distance along a transmission line (Chapter XIV). / Any constant. / A positive integer. }

$H(\lambda) = A_n\lambda^k + A_{n-2}\lambda^{n-2} + \cdots + A_0$ (Chapter VII).

H_{MN} = Voltage at point N due to unit suddenly-impressed voltage at M (Chapter XV).

$i = i(t)$ = { Current. / Current in a line at distance x from generator end. }

i_D = Current at far end of line.

$i_k(t)$ = Steady-state current in Circuit k.

$i_0(t)$ = { Current at generator end of line. / Current in circuit if variable parameter were absent. }

$i_r(t)$ = Current in rotor circuit.

$i_s(t)$ = { Current in stator circuit. / Steady-state component of current. }

Functions of time.

I = { Current at $t = 0$. / A constant current. / Peak value of an alternating current. / Peak current at distance x from generator end of line. }

I_k = { Value of current in Circuit k at $t = 0$. / Peak value of an alternating current in Circuit k. }

I_0 = Current at generator end of line.

Not functions of time.

$j = \sqrt{-1}$.

$J_n(x)$ = Bessel function of the first kind and nth order.

$J_n(jx)$ = Bessel function of the first kind and nth order of imaginary argument.

k = { Any circuit of a network. / A constant. / A positive half-integer. }

K = A constant.

K_{kh} = An integration constant (k identifies a current, and h a root).

$K_n(x)$ = Bessel function of the second kind and nth order.

l = Inductance per unit length of line.
L = Inductance (circuit with lumped parameters).
L_0 = A terminal inductance.
L_{hh} = Self-inductance of the h Circuit.
L_{hk} = Mutual inductance of the h and k Circuits, voltage produced in the h Circuit due to unit rate of change of current in the k Circuit.
m = { A constant. / Any real number.
$m!$ = $1 \cdot 2 \cdot 3 \cdot \cdots m$, factorial m.
$M_{kk}(p)$ = Minor of the hth row and kth column of $D(b)$. Also the polynomial obtained upon simplifying the minor.
ma. = Milliampere.
mfd. = Microfarad.
n = { The last circuit of a network. / A constant.
N = A general fixed-parameter network.
p = The fundamental time operator. Equivalent to $\frac{d}{dt}$ when applied to a continuous function.
p^{-1} = The operator which is the inverse of p.
P = Any point in a network.
$P(\omega)$ = A function giving the variation with frequency of the power component of current for unit alternating e.m.f.
q = $q(t)$ = Electric charge, a function of time.
Q = Electric charge, a constant.
$Q(\omega)$ = A function giving the variation with frequency of the quadrature component of current for unit alternating e.m.f.
r = { Resistance per unit length of line. / A radius.
r_0 = A constant.
R = Resistance (circuit with lumped parameters).
$\overline{R}$ = Equivalent resistance (Chapter VII).

R_{hh} = Total resistance of the h Circuit.

R_{hk} = Mutual resistance of the h and k Circuits, voltage produced in Circuit h by unit current in Circuit k.

$\mathfrak{R}$ = Residue of a function at a pole.

s = { Distance along a curve. / Half-period of oscillation.

S = { A switch. / A spring.

t = Time.

u = A variable.

v = { Volts. / $\frac{1}{\sqrt{lc}}$, velocity of propagation.

$V_{hh}(t)$ = Indicial impedance, voltage due to unit current suddenly impressed.

$V_{hk}(t)$ = Transfer indicial impedance, voltage in Circuit h due to unit current suddenly applied in Circuit k.

$\overline{V}(t)$ = Finite part of the indicial impedance, $V(t)$ less pulse.

x = { Distance along a transmission line. / Any variable.

$|x|$ = Absolute value of x.

y = A variable.

z = Impedance.

$z_{hh} = R_{hh} + L_{hh}p + \frac{1}{C_{hh}p}$, the self-impedance of the h Circuit of a network.

$z_{hk} = R_{hk} + L_{hk}p + \frac{1}{C_{hk}p}$, mutual impedance between the h and k Circuits.

$z_0(p) = \sqrt{\frac{r + lp}{g + cp}}$, the characteristic impedance of a transmission line.

$Z_{hk}(p) = \frac{D(p)}{M_{hk}(p)}$, the impedance function.

$\frac{1}{Z(p)}$ = Operator formed from the network impedance, or more generally from the alternating-current steady-state proportionality factor between cause and effect.

α = A decrement (the real component of a complex constant).

α_k = Decrement of the k Mode.

$\overline{\alpha_k}$ = Approximate decrement of the k Mode.

$\alpha_0 = \sqrt{r_0 c_0}$, a constant in Chapter XIV.

β = Attenuation constant of a line,

$$\beta = \sqrt{(r + lp)(g + cp)} \text{ in general.}$$

$\gamma = rcx^2$.

$\Gamma(n)$ = The gamma function of n.

Δ = The fundamental space operator. Equivalent to $\frac{d}{dx}$ when applied to a continuous function.

$\Delta\lambda$ = An increment in λ.

ϵ = The Naperian base, 2.71828 . . . .

ζ = A variable of integration (Chapter XII).

θ = { Angle between zero of voltage wave and zero time. / Any angle.

λ = { Generalized angular velocity $= \alpha + j\omega$. / A variable of integration.

ξ = A variable of integration, Chapter X.

π = 3.14159 . . . .

ρ = { $\frac{r}{2l} + \frac{g}{2c}$, an attenuation constant. / The magnitude of a radius vector.

$\sigma = \frac{r}{2l} - \frac{g}{2c}$, a distortion constant.

$\sigma(j\omega)$ = Alternating-current terminating impedance at distant end of line.

ϕ = { Phase angle between voltage and current. / Any angle.

$\phi(t)$ = A function of t.
$\psi(t)$ = A function of t.
ω = $2\pi f$, the angular velocity.
Ω = ohms.
ω_k = Angular velocity of the k Mode.
$\bar{\omega}_k$ = Approximate angular velocity of the k Mode.
1 = The unit function (zero when $t < 0$, unity when $t > 0$).
1_h = The discontinuous function which is zero when $t < h$ and unity when $t > h$.
∞ = Infinity.

APPENDIX B

FOURIER ANALYSIS AND ASYMPTOTIC SERIES

BY NORBERT WIENER

Integration. The integral between a and b of the continuous function $f(t)$ is defined in the usual way:

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\Big(a + \frac{k}{n}(b-a)\Big). \qquad (1)$$

We then have:

if $$f(t) = \phi_1(t) + \phi_2(t)$$

then $$\int_a^b f(t)\,dt = \int_a^b \phi_1(t)\,dt + \int_a^b \phi_2(t)\,dt. \qquad (2)$$

This theorem cannot, however, be applied to infinite sums for if

$$f_m(t) = \phi_1(t) + \phi_2(t) + \cdots + \phi_m(t)$$

and $$f(t) = \lim_{m\to\infty} f_m(t)$$

exists (even as a bounded limit) and

$$\lim_{m\to\infty} \int_a^b f_m(t)\,dt$$

exists, it does not follow that $f(t)$ has an integral in the sense already defined.

As an example, let $f(t) = 0$ except between $\frac{1}{n} - \frac{1}{n^4}$ and $\frac{1}{n} + \frac{1}{n^4}$, where

$$f(t) = n\Big(1 - n^4\Big|t - \frac{1}{n}\Big|\Big), \qquad n = 2, 3 \ldots.$$

$n - n^5\left|t - \frac{1}{n}\right|$

Clearly $f(t)$ will not possess an integral in the sense already given. On the other hand, if

$$f_m(t) = f(t) \qquad \text{for } t > \frac{1}{m} + \frac{1}{m^4}$$

and 0 otherwise, then $f_m(t)$ will be continuous, and the conditions stated will hold.

Now, probabilities and integrals are related in the very closest possible way, and it is frequently useful in the theory of probabilities and elsewhere — in statistical mechanics, for example — to add an infinite set of integrals and to obtain a single integral. This obviously demands an extension of the notion of integral, and fortunately this extension can be carried through in just one way. In other words, there is just one operator $\int_a^b f(t)\,dt$ which yields for continuous functions the result already defined, and which has the property that if the set of functions $\{f_m(t)\}$ contains no function with absolute value exceeding A, and

$$\lim_{m\to\infty} f_m(t) = f(t), \qquad a \leq t \leq b$$

then

$$\lim_{m\to\infty} \int_a^b f_m(t)\,dt$$

exists and equals $\int_a^b f(t)\,dt$.

This definition yields, as far as bounded functions are concerned, the integral of Lebesgue, discovered by Lebesgue in 1900.[1] It is discussed in detail in such books as Vol. I of Hobson's Theory of Functions of a Real Variable, and is indeed given a quite different definition. We are, however, interested here in stating its properties rather than in deducing them.

[1] H. Lebesgue, Leçons sur l'Intégration, Paris, Gauthier-Villars.

If $f(t)$ is not bounded, let

$$f_{A,B}(t) \begin{cases} = f(t), & A \leq f(t) \leq B, \\ = A, & f(t) < A, \\ = B, & f(t) > B \end{cases} \tag{3}$$

If $\int_a^b f_{A,B}(t)\, dt$ approaches a unique limit as $A \to -\infty$ and $B \to \infty$ in any way, this limit is again called $\int_a^b f(t)\, dt$ and is known as the Lebesgue integral. In what follows all integrals are taken in the sense of Lebesgue.

2. **Fourier Coefficients.** Let $f(t)$ be any function (possibly complex valued) of a real variable t, defined for

$$-\pi \leq t \leq \pi,$$

possessing a Lebesgue integral, and such that the square of its modulus possesses a Lebesgue integral. Let us consider

$$I_n = \int_{-\pi}^{\pi} \left| f(t) - \sum_{-n}^{n} \frac{a_k}{\sqrt{2\pi}} \epsilon^{jkt} \right|^2 dt \tag{4}$$

which may be regarded in the least square sense as a measure of the amount by which

$$\sum_{-n}^{n} \frac{a_k}{\sqrt{2\pi}} \epsilon^{jkt}$$

fails to equal $f(t)$. We ask, what values of a_k minimize I? By elementary methods of calculus, we can show that these values are

$$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, \epsilon^{-jkt}\, dt. \tag{5}$$

These quantities are known as the **Fourier coefficients** of $f(t)$.

As we have just seen, if we replaced a_n and a_{-n} by zero instead of the values (5), we increase I_n or leave it unchanged. Hence if we give the a_k's the values in (5), we have

$$I_1 \geq I_2 \geq I_3 \cdots \geq I_n \geq \cdots \geq 0.$$

This suggests the possibility that

$$\lim_{n\to\infty} I_n = 0$$

and this is actually the case. The theorem to this effect is due to Hurwitz,[1] and takes his name.

It follows from the Hurwitz theorem that

$$\int_{-\pi}^{\pi} |\, f(t)\,|^2\, dt = \sum_{-\infty}^{\infty} |\, a_k\,|^2 \tag{6}$$

or more generally, that if a_k is defined as above, while $g(t)$ satisfies the conditions laid down for $f(t)$ and

$$b_k = \int_{-\pi}^{\pi} g(t)\, \epsilon^{-jkt}\, dt,$$

then

$$\int_{-\pi}^{\pi} f(t)\, \bar{g}(t)\, dt = \sum_{-\infty}^{\infty} a_k \bar{b}_k, \tag{7}$$

the bar denoting the conjugate complex quantity of the quantity over which it is placed. Theorems (6) and (7) are known as the Parseval theorems.

An important question is whether Hurwitz's theorem possesses a valid inverse. It does not possess a valid inverse unless the integration involved is of the Lebesgue type. This inverse, known as the Riesz-Fischer theorem,[2] reads:

Let $\sum_{-\infty}^{\infty} |\, a_k\,|^2$ converge. Then there is a function $f(t)$, such that

$$\int_{-\pi}^{\pi} f(t)\, dt \quad \text{and} \quad \int_{-\pi}^{\pi} |\, f(t)\,|^2\, dt$$

[1] Hurwitz, Math. Annalen, Vol. LVII, 1903, p. 426.

[2] F. Riesz, Sur les Systèmes orthogonaux de Fonctions, Comptes Rendus, Vol. 144, 1907, pp. 615–619.

E. Fischer, Sur la convergence en moyenne, *Ibid.*, pp. 1022–1024.

exist, and such that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(t) - \sum_{-n}^{n} \frac{a_k}{\sqrt{2\pi}} \epsilon^{jkt} \right|^2 dt = 0. \tag{8}$$

Formula (5) will hold, of course.

The following remarks are in order as to the Riesz-Fischer theorem:

(*a*) It forms the chief justification for the use of the Lebesgue integral. No less general integral is adequate to the needs of harmonic analysis.

(*b*) The function $f(t)$ is not uniquely determined. It may be arbitrarily modified, for example, at all rational points. If, however, two functions $\phi_1(t)$ and $\phi_2(t)$ are considered as **equivalent** if

$$\int_{-\pi}^{\pi} | \phi_1(t) - \phi_2(t) |^2 \, dt = 0, \tag{9}$$

then all functions $f(t)$ are equivalent. Two functions equivalent to a third function are equivalent to each other, and two equivalent **continuous** functions are identical.

3. **Fourier Duality.** Let it be noted that the relations already mentioned may be written in pairs as follows:

$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) \epsilon^{-jkt} \, dt$	$f(t)$ approached in least-square sense by partial sums of $\frac{1}{\sqrt{2\pi}} \sum a_k \epsilon^{jkt}$
$\int_{-\pi}^{\pi} \| f(t) \|^2 \, dt$	$= \sum_{-\infty}^{\infty} \| a_k \|^2$
$\int_{-\pi}^{\pi} f(t) \bar{g}(t) \, dt$	$= \sum_{-\infty}^{\infty} a_k \bar{b}_k$

The expressions on the right correspond to those on the left under the following transformations:

Subscript k corresponds to argument t

a_k, b_k correspond to $f(t)$, $g(t)$

$\sum_{-\infty}^{\infty}$ corresponds to $\int_{-\pi}^{\pi} dt$

ϵ^{jkt} corresponds to ϵ^{-jkt}

If we replace the interval $(-\pi, \pi)$ by the interval $(-A,A)$, it is easy to see that our duality takes the form:

$a(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(t)\, \epsilon^{-j\mu t}\, dt$	$f(t)$ approached in least-square sense by partial sums of $\frac{1}{\sqrt{2\pi}} \sum_{\frac{A\mu}{\pi}=-\infty}^{\frac{A\mu}{\pi}=\infty} a(\mu)\epsilon^{j\mu t}\left(\frac{\pi}{A}\right)$
$\int_{-A}^{A} \mid f(t) \mid^2 dt$	$= \sum_{\frac{A\mu}{\pi}=-\infty}^{\frac{A\mu}{\pi}=\infty} \mid a(\mu) \mid^2 \left(\frac{\pi}{A}\right)$
$\int_{-A}^{A} f(t)\bar{g}(t)\, dt$	$= \sum_{\frac{A\mu}{\pi}=-\infty}^{\frac{A\mu}{\pi}=\infty} a(\mu)\bar{b}(\mu)\left(\frac{\pi}{A}\right)$

where $\mu = \frac{k\pi}{A}$.

Note that

μ corresponds to t

$\frac{\pi}{A} \sum_{\frac{A\mu}{\pi}=-\infty}^{\frac{A\mu}{\pi}=\infty}$ corresponds to $\int_{-A}^{A} dt$

$a(\mu), b(\mu)$ corresponds to $f(t), g(t)$

$\epsilon^{j\mu t}$ corresponds to $\epsilon^{-j\mu t}$

The range of μ is from $-\infty$ to ∞ by steps of the size $\frac{\pi}{A}$. The range of t is continuously from $-A$ to A. As $A \to \infty$, these ranges both tend in different ways towards the con-

tinuous range $(-\infty, \infty)$. In the limit we get formally

$$a(\mu) \backsim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \epsilon^{-j\mu t}\, dt \qquad f(t) \backsim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\mu) \epsilon^{j\mu t}\, d\mu$$

$$\int_{-\infty}^{\infty} |\, f(t)\, |^2\, dt \qquad = \int_{-\infty}^{\infty} |\, a(\mu)\, |^2\, d\mu$$

$$\int_{-\infty}^{\infty} f(t) \bar{g}(t) \qquad = \int_{-\infty}^{\infty} a(\mu) \bar{b}(\mu)\, d\mu$$

We understand by $\backsim$ some correspondence later to be specified. Here perfect symmetry and reciprocality is obtained, barring the change of the sign of the exponent of $\epsilon^{j\mu t}$. How can the conditions for this equivalence be specified?

4. **The Plancherel Theorem.** In the theory of Fourier series we have seen that there is a certain duality between the Hurwitz theorem and the Riesz-Fischer theorem. Nevertheless the Hurwitz theorem may be treated on the basis of a theory of continuous functions alone, while the Riesz-Fischer theorem involves Lebesgue integration. This discrepancy is due to the asymmetry of the ordinary Fourier series theory, in which to a function of a real variable there corresponds that much simpler thing, an array of coefficients. In the case where t runs over an infinite range, the coefficients and the function assume the same limiting form — as did the tortoise and the hedgehog in Kipling's Just-So explanation of the origin of the armadillos. In this infinite case the quasi-Hurwitz theorem becomes identical with the quasi-Riesz-Fischer theorem, and involves the same apparatus of Lebesgue integration and least-square approximation which we have found to characterize the simple Riesz-Fischer theorem. This is perhaps the explanation of the much later growth of the infinite or Fourier integral theory, as opposed to that of Fourier series. The fundamental theorem is due to Plancherel[1] and reads as follows:

[1] M. Plancherel, Contribution à l'étude de la Représentation d'une fonction arbitraire par des integrales definies. Rend. di Palermo, Vol. 30, 1910, pp. 289–335.

Let $f(t)$ be a function such that

$$\int_{-A}^{A} f(t)\, dt$$

exists in the Lebesgue sense for every A, while

$$\lim_{A\to\infty} \int_{-A}^{A} |\, f(t)\, |^2\, dt$$

exists as a finite limit of a Lebesgue integral. Then there is a function $a(\mu)$ such that

$$\lim_{A\to\infty} \lim_{B\to\infty} \int_{-B}^{B} \left| a(\mu) - \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(t)\epsilon^{-j\mu t}\, dt \right|^2 d\mu = 0. \quad (10)$$

The integrals

$$\int_{-A}^{A} a(\mu)\, d\mu \quad \text{and} \quad \int_{-A}^{A} |\, a(\mu)\, |^2\, d\mu$$

exist in the Lebesgue sense for every A, while

$$\lim_{A\to\infty} \int_{-A}^{A} |\, a(\mu)\, |^2\, d\mu = \lim_{A\to\infty} \int_{-A}^{A} |\, f(t)\, |^2\, dt \quad (11)$$

Furthermore,

$$\lim_{A\to\infty} \lim_{B\to\infty} \int_{-B}^{B} |\, f(t) - \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} a(\mu)\epsilon^{j\mu t}\, d\mu\, |^2\, dt = 0. \quad (12)$$

Formulae (10) and (12) give us the proper interpretation for the symbol $\backsim$.

The function $a(\mu)$ is known as the **Fourier transform** of $f(t)$. It is so determined that if $a_1(\mu)$ is another possible Fourier transform of $f(t)$,

$$\int_{-A}^{A} |\, a_1(\mu) - a(\mu)\, |^2\, d\mu = 0 \quad (13)$$

for every A. A function can have only one continuous Fourier transform, if any.

The Parseval theorem for Fourier transforms reads as follows:

Let $f(t)$ and $g(t)$ fulfill the conditions laid down for $f(t)$ in the hypothesis to the Plancherel theorem, and let $a(\mu)$ and $b(\mu)$ be their Fourier transforms. Then

$$\lim_{A,B\to\infty}\int_{-A}^{B} f(t)\,\bar{g}(t)\,dt = \lim_{A,B\to\infty}\int_{-A}^{B} a(\mu)\bar{b}(\mu)\,d\mu. \qquad (14)$$

5. **Local and Asymptotic Properties.** In a generalized sense we may say that the Fourier transform of a function of period 2π is a function defined only for integral arguments. Periodicity is a property at infinity or an asymptotic property, while the property of being defined for only integral arguments is a local property, as it involves each integral argument separately. This is a crude illustration of a duality running through the whole theory of Fourier analysis.

Again, let us write formally for the $f(t)$ in terms of its Fourier transform $a(\mu)$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\mu)\epsilon^{j\mu t}\,d\mu. \qquad (15)$$

On differentiating formally,

$$f'(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} j\mu a(\mu)\epsilon^{j\mu t}\,d\mu. \qquad (16)$$

That is, the derivative of $f(t)$ is the Fourier transform of $j\mu a(\mu)$. The condition

$$\int_{-\infty}^{\infty} |\,f'(t)\,|^2\,dt \qquad \text{finite}$$

is primarily a local, differential condition on $f(t)$; the condition

$$\int_{-\infty}^{\infty} \mu^2\,|\,a(\mu)\,|^2\,d\mu \qquad \text{finite}$$

is an asymptotic condition. All of this can be made rigorous with the proper apparatus of assumptions.

More generally, we define $F\left(\frac{d}{dt}\right)f(t)$ as the function with the Fourier transform $F(j\mu)a(\mu)$, at least when F is a polynomial. While this is presented here formally, it may be put on a sound basis. This method furnishes one of the most useful approaches to the operatioual calculus.

6. **Retrospective or Historical Operators.** An interesting question is: What is the Fourier transform of a function vanishing for negative arguments? Such a transform will be represented as the least-square limit as $A \to \infty$ of

$$a(\mu) = \frac{1}{\sqrt{2\pi}} \int_0^A f(t)\epsilon^{-j\mu t}\, dt \tag{17}$$

where f possesses the conditions already laid down for it to possess a Fourier transform. The necessary and sufficient additional condition imposed on $a(\mu)$ reads as follows: No condition is imposed on the real and the pure imaginary parts of $a(\mu)$ separately, as may be seen on inspection, but together they must constitute the real and the imaginary parts of the boundary values of a function analytic in the lower half-plane. Of course this needs further mathematical specification to be made precise, but its general meaning can be seen from the fact that the integral

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\epsilon^{-\lambda t - j\mu t}\, dt$$

converges for $\lambda > 0$.

It follows readily from this that if $F(j\mu)$ is a function of μ admitting of a Fourier transform, analytic below the axis of reals and with certain restrictions as to its behavior at infinity, which in effect prevent the existence of a source there, $F(j\mu)a(\mu)$ enjoys the same property as does $a(\mu)$, of being the Fourier transform of a function vanishing to the left of the origin. That is, $F\left(\frac{d}{dt}\right)f(t)$ will vanish to the left of the origin. If $F(j\mu)$ is bounded in the lower half-

plane, all necessary restrictions as to its behavior at infinity are fulfilled.

It is possible to reverse the argument and to deduce from this that such an operator $F\left(\frac{d}{dt}\right)$ only depends on the history of a function, inasmuch as two functions with the same history up to a given point have a difference vanishing up to this point.

7. **The Separation of a Function into High- and Low-Frequency Ranges.** If t is taken as the time in the above formulae, μ stands for an angular velocity. In the discussion of functions of time, from the standpoint of the theory of Fourier transforms, there are certain difficulties which belong peculiarly to the high-frequency range of the functions considered, and certain quite distinct difficulties which belong to the low-frequency range. In general it is the high-frequency terms in the harmonic analysis of a function which determine its properties of continuity, while the low-frequency terms are more concerned with its behavior at infinity. It is possible in many cases to separate a function into low-frequency and high-frequency parts and thus to secure a simplification of our problems.

As a useful tool in this separation let us introduce the notion of a non-analytic, infinitely differentiable function. An example is

$$\left.\begin{aligned} \phi_{A,B}(\mu) &= 1, \quad |\mu| < A \\ &= 0, \quad |\mu| > B \\ &= \frac{\int_A^{\mu} \epsilon^{\frac{1}{(\xi-A)(\xi-B)}}\, d\xi}{\int_A^{B} \epsilon^{\frac{1}{(\xi-A)(\xi-B)}}\, d\xi}, \quad A \leq |\mu| \leqq B \end{aligned}\right\} \tag{18}$$

We can represent any frequency function $a(\mu)$ as the sum of two parts; a low-frequency part, $a(\mu)\phi_{A,B}(\mu)$, and a high-frequency part, $a(\mu)\,(1 - \phi_{A,B}(\mu))$. If $a(\mu)$ is the Fourier transform of $f(t)$ and $\phi_{A,B}(\mu)$ is the Fourier transform of

$\Phi_{A,B}(t)$, this separation of a into two parts corresponds to the separation of $f(t)$ into the two parts:

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} a(\mu)\phi_{A,B}(\mu)\epsilon^{j\mu t}\, d\mu \tag{19}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\Phi_{A,B}(\tau + t)\, d\tau \tag{20}$$

and

$$f_2(t) = f(t) - f_1(t), \tag{21}$$

which we may call the low- and high-frequency parts of $f(t)$, respectively. Now, it is easy to show that for any k

$$\lim_{t\to\pm\infty} \Phi_{A,B}(t)t^k = 0, \tag{22}$$

while $\Phi_{A,B}$ is a continuous function with derivatives of all orders. Hence even if $f(t)$ does not possess a Fourier transform, but has with its square a Lebesgue integral over every finite range, and is of finite order at $\pm\infty$, the division into $f_1(t)$ and $f_2(t)$ will still hold valid.

8. **Asymptotic Series.** A case of particular interest is where

$$f(t) = 0, \quad t < 0$$
$$= 1, \quad t > 0$$

that is, the Heaviside *1* function. Here $f_1(t)$ will be of the form shown in Fig. 1 and will be analytic, while $f_2(t)$ will be

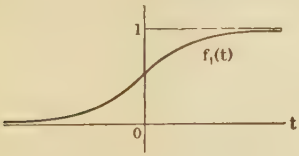

FIG. 1

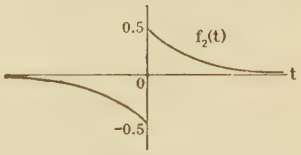

FIG. 2

of the form shown in Fig. 2 and will vanish at $\pm\infty$ faster than any negative power of t. This suggests that we may divide the problem of defining

$$F\left(\frac{d}{dt}\right)f(t)$$

into two parts, one for the high-frequency terms and one for the low-frequency terms, and separate those difficulties which come from the singularity of $f(t)$ at the origin from those which arrive from its behavior at infinity.

We shall consider the particular case where $F(j\mu)$ possesses the following properties:

(*1*) There is a neighborhood of the origin u within which $F(\mu^2)$ assumes the values of an analytic function of μ.

(*2*) $F(j\mu)$ is of the form *constant* + $g(\mu)$ for μ = real, where $g(\mu)$ is a function with a Lebesgue integral over every finite range, and is such that

$$\lim_{A\to\infty} \int_{-A}^{A} |\, g(\mu)\,|^2\, d\mu$$

exists. All the successive derivatives of $F(j\mu)$ exist and are of the form $g(\mu)$, under the same conditions. We can then show that

(*a*) A and B may be so chosen that if we put

$$F(j\mu) = \sum a_n (j\mu)^{\frac{n}{2}}, \tag{23}$$

in the neighborhood of the origin, then the series

$$\sum a_n \left(\frac{d}{dt}\right)^{\frac{n}{2}} f_1(t)$$

converges, and has after $2k+1$ terms a remainder which is of the order of magnitude of $t^{-(k+\frac{1}{2})}$ at infinity. Here we define

$$\left(\frac{d}{dt}\right)^{\frac{2k+1}{2}} h(t) = \left(\frac{d}{dt}\right)^{-\frac{1}{2}} \left(\frac{d}{dt}\right)^{k+1} h(t)$$

where

$$\left(\frac{d}{dt}\right)^{-\frac{1}{2}} \psi(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{\psi(\tau)}{\sqrt{t-\tau}}\, d\tau.$$

We can show that

(*b*) $$\lim_{t\to\infty} \left(\frac{d}{dt}\right)^{\frac{2k+1}{2}} f_1(t) - \frac{(-\frac{1}{2})^k (1\cdot 3\cdot 5\cdot \cdots 2k-1)}{\sqrt{\pi} t^{\frac{2k+1}{2}}} = 0 \tag{24}$$

and that

$$\lim_{t\to\infty}\left(\frac{d}{dt}\right)^k f_1(t) = 0. \tag{25}$$

So far only condition (*1*) has come into consideration. For $f_1(t)$, let us then put by definition

$$F\left(\frac{d}{dt}\right)f_1(t) = \sum a_n\left(\frac{d}{dt}\right)^{\frac{n}{2}} f_1(t). \tag{26}$$

As to $f_2(t)$, it has a Fourier transform, and we define $F\left(\frac{d}{dt}\right)f_2(t)$ as before. We can show that under our hypotheses,

$$(c) \qquad \lim_{t\to\infty} \quad t^k F\left(\frac{d}{dt}\right)f_2(t) = 0 \tag{27}$$

for all k.

If we put

$$F\left(\frac{d}{dt}\right)f(t) = F\left(\frac{d}{dt}\right)f_1(t) + F\left(\frac{d}{dt}\right)f_2(t) \tag{28}$$

we can show that this function is independent of the auxiliary numbers A and B chosen. Theorems (*a*), (*b*), and (*c*) then give

$$F\left(\frac{d}{dt}\right)f(t) - F(0) + \frac{a_1}{2\sqrt{\pi t}} - \sum_1^n \frac{a_{2k+1}(-\frac{1}{2})^k(1\cdot 3\cdot 5\cdot\cdots 2k-1)}{\sqrt{\pi}\, t^{\frac{2k+1}{2}}}$$

$= $ expression of order of magnitude of $t^{-\frac{2n+3}{2}}$ at $+\infty$. This is, however, what we mean when we say that the asymptotic development of

$$F\left(\frac{d}{dt}\right)f(t)$$

is

$$F(0) - \frac{a_1}{2\sqrt{\pi t}}t + \sum_1^\infty \frac{a_{2k+1}(-\frac{1}{2})^k(1\cdot 3\cdot 5\cdot\cdots 2k-1)}{\sqrt{\pi}\, t^{\frac{2k+1}{2}}}. \tag{29}$$

APPENDIX C

TABLE OF OPERATIONAL FORMULAS

The number in parenthesis refers to an equation in the text where a similar operational formula is developed.

1. $\frac{1}{p^m}\,1 = \frac{t^m}{m!}$ (205)

2. $\frac{1}{p+b}\,1 = \frac{1}{b}(1 - \epsilon^{-bt})$ (212)

3. $\frac{p}{p+b}\,1 = \epsilon^{-bt}$ (218)

4. $\frac{\omega p}{p^2+\omega^2}\,1 = \sin \omega t$ (221)

5. $\frac{p^2}{p^2+\omega^2}\,1 = \cos \omega t$ (223)

6. $\frac{\omega^2}{p^2+\omega^2}\,1 = 1 - \cos \omega t$ (222)

7. $\frac{1}{(p+b)^2}\,1 = \frac{1}{b^2}[1 - \epsilon^{-bt}(1 + bt)]$

8. $\frac{p}{(p+b)^2}\,1 = t\epsilon^{-bt}$ (251)

9. $\frac{p}{(p+b)^n}\,1 = \frac{t^{n-1}}{(n-1)!}\,\epsilon^{-bt}$ (252)

10. $\frac{p^2}{(p+b)^2}\,1 = (1 - bt)\,\epsilon^{-bt}$

11. $\frac{1}{(p+a)(p+b)}\,1 = \frac{1}{ab} + \frac{1}{b-a}\left(\frac{\epsilon^{-bt}}{b} - \frac{\epsilon^{-at}}{a}\right)$

12. $\frac{p}{(p+a)(p+b)}\,1 = \frac{1}{a-b}(\epsilon^{-bt} - \epsilon^{-at})$

13. $\frac{p^2}{(p+a)(p+b)}\,1 = \frac{1}{b-a}(b\epsilon^{-bt} - a\epsilon^{-at})$

14. $\frac{p}{(p^2+\omega^2)(p+b)}\,1 = \frac{\epsilon^{-bt}}{b^2+\omega^2} + \frac{1}{\omega\sqrt{b^2+\omega^2}}\sin(\omega t - \theta)$,

where $\theta = \tan^{-1}\frac{\omega}{b}$

15. $\dfrac{p^2 + 2\omega^2}{p^2 + 4\omega^2}\, 1 = \cos^2 \omega t$ (291)

16. $\left(\dfrac{bp - 1}{bp + 1}\right)^2 1 = \dfrac{1}{b}\,(b - 4\, t\epsilon^{-\frac{1}{b}t})$

17. $\dfrac{p}{ap^2 + bp + c}\, 1 = \dfrac{1}{a\sqrt{\dfrac{c}{a} - \dfrac{b^2}{4\,a^2}}}\, \epsilon^{-\frac{bt}{2\,a}} \sin \sqrt{\dfrac{c}{a} - \dfrac{b^2}{4\,a^2}}\, t$ (286)

18. $\dfrac{\omega p}{(p + b)^2 + \omega^2}\, 1 = \epsilon^{-bt} \sin \omega t$ (285)

19. $\dfrac{\omega p}{(p + b)^2 - \omega^2}\, 1 = \epsilon^{-bt} \sinh \omega t$ (323)

20. $\dfrac{p^2 + bp}{(p + b)^2 + \omega^2}\, 1 = \epsilon^{-bt} \cos \omega t$ (287)

21. $p\epsilon^{-bt}\, 1 = -b\epsilon^{-bt}\, 1 + p\, 1$ (233)

22. $p\, f(t)\, 1 = f'(t)\, 1 + f(0)\, p\, 1$ (267)

23. $p \cos \omega t\, 1 = -\omega \sin \omega t\, 1 + p\, 1$ (237)

24. $\dfrac{p^2}{p + b}\, 1 = -b\epsilon^{-bt}\, 1 + p\, 1$ (254)

25. $\dfrac{p}{p + b}\, t\epsilon^{-bt}\, 1 = \left(1 - \dfrac{bt}{2}\right) t\epsilon^{-bt}$

26. $\dfrac{p^2 + \omega^2}{p\omega} \sin \omega t\, 1 = 1$ (241)

27. $\epsilon^{-bt}\, \dfrac{1}{Z(p)}\, 1 = \dfrac{1}{Z(p + b)}\, \epsilon^{-bt}\, 1$ (274)

28. $\dfrac{1}{Z(p)}\, \epsilon^{-bt}\, 1 = \epsilon^{-bt}\, \dfrac{1}{Z(p - b)}\, 1$ (283)

29. $\dfrac{1}{Z(p)}\, 1 = \epsilon^{-bt}\, \dfrac{p}{p - b}\, \dfrac{1}{Z(p - b)}\, 1$ (280)

30. $\dfrac{1}{Z(p)}\, \epsilon^{-bt} f(t)\, 1 = \epsilon^{-bt}\, \dfrac{1}{Z(p - b)}\, f(t)\, 1$ (293)

31. $\sin \omega t\, \dfrac{1}{Z(p)}\, 1 = \dfrac{p}{2\,j}\left(\dfrac{1}{(p - j\omega)Z(p - j\omega)} - \dfrac{1}{(p + j\omega)Z(p + j\omega)}\right) 1$ (289)

32. $\cos \omega t\, \dfrac{1}{Z(p)}\, 1 = \dfrac{p}{2}\left(\dfrac{1}{(p - j\omega)Z(p - j\omega)} + \dfrac{1}{(p + j\omega)Z(p + j\omega)}\right) 1$ (288)

33. $\epsilon^{-bp}\,1 = 1_b$ (310)

34. $\epsilon^{-bp} f(t)\,1 = f(t-b)\,1_b$ (305)

35. $\epsilon^{-bp}\dfrac{1}{Z(p)}\,1 = f(t-b)\,1_b$ (309)

36. $\epsilon^{-vt\frac{d}{dx}} f(x) = f(x-vt)$ (315)

37. $\tanh\dfrac{bp}{2}\,1 = \dfrac{4}{\pi}\sum_1^\infty \dfrac{1}{h}\sin\dfrac{h\pi t}{b}$, h odd (368) and (373)

38. $p^{\frac{1}{2}}\,1 = \dfrac{1}{\sqrt{\pi t}}$ (446)

39. $p^{-\frac{1}{2}}\,1 = \dfrac{2}{\sqrt{\pi}}\sqrt{t}$ (452)

40. $p^{\frac{3}{2}}\,1 = -\dfrac{1}{2\sqrt{\pi}}t^{-\frac{3}{2}}$ (462)

41. $p^m\,1 = \dfrac{t^{-m}}{\Gamma(1-m)}$, except for m a positive integer (463)

42. $p^m t^n\,1 = \dfrac{\Gamma(n+1)t^{n-m}}{\Gamma(n-m+1)}$, provided the gamma functions exist (456)

43. $p^{\frac{1}{2}} f(t)\,1 = \dfrac{1}{\sqrt{\pi}}\dfrac{d}{dt}\displaystyle\int_0^t \dfrac{f(\lambda)}{\sqrt{t-\lambda}}\,d\lambda$ (512)

44. $p^m f(t)\,1 = \displaystyle\sum_0^\infty a_n \dfrac{\Gamma(n+1)t^{n-m}}{\Gamma(n-m+1)}$, provided $f(t) = \displaystyle\sum_0^\infty a_n t^n$, and the gamma functions exist (513)

45. $p\epsilon^{-\sqrt{bp}}\,1 = \dfrac{1}{2}\sqrt{\dfrac{b}{\pi}}\,t^{-\frac{3}{2}}\epsilon^{-\frac{b}{4t}}$ (509)

46. $\dfrac{p}{\sqrt{p^2+b^2}}\,1 = J_0(bt)$ (473)

47. $\dfrac{p}{\sqrt{p^2-b^2}}\,1 = J_0(jbt)$ (474)

48. $\dfrac{p}{\sqrt{p^2+b^2}}\left(\dfrac{\sqrt{p^2+b^2}-p}{b}\right)^n 1 = J_n(bt)$ (476)

49. $\epsilon^{-\frac{b}{p}}1 = J_0(2\sqrt{bt})$ (330)

50. $\dfrac{1}{\sqrt{p+b}}\,1 = \displaystyle\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{\pi\lambda}}\,d\lambda$ (483)

51. $\dfrac{1}{\sqrt{p+b}}\,1 = 2\sqrt{\dfrac{t}{\pi}}\,\epsilon^{-bt}\left(1+\dfrac{2\,bt}{1\cdot 3}+\dfrac{(2\,bt)^2}{1\cdot 3\cdot 5}+\cdots\right)$ (484)

52. $\dfrac{p}{\sqrt{p+b}}\,1 = \epsilon^{-bt}\,\dfrac{1}{\sqrt{\pi t}}$ (482)

53. $\dfrac{p}{\sqrt{p+b}}\,\epsilon^{-bt}\sqrt{t}\,1 = \dfrac{\sqrt{\pi}}{2}\,\epsilon^{-bt}(1-bt)$

54. $\sqrt{p+b}\,1 = \epsilon^{-bt}\,\dfrac{1}{\sqrt{\pi t}} + b\displaystyle\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{\pi\lambda}}\,d\lambda$ (486)

55. $\dfrac{1}{\sqrt{1+\dfrac{2\,b}{p}}}\,1 = \epsilon^{-bt}J_0(jbt)$ (478)

56. $\dfrac{1}{1+\sqrt{\dfrac{b}{p}}}\,1 = \epsilon^{bt}\left(1-\sqrt{\dfrac{b}{\pi}}\displaystyle\int_0^t \frac{\epsilon^{-b\lambda}}{\sqrt{\lambda}}\,d\lambda\right)$ (497)

57. $\dfrac{1}{1+\sqrt{\dfrac{b}{p}}}\,1 = \epsilon^{bt} - 2\sqrt{\dfrac{bt}{\pi}}\left(1+\dfrac{2\,bt}{1\cdot 3}+\dfrac{(2\,bt)^2}{1\cdot 3\cdot 5}+\cdots\right)$ (498)

58. $\dfrac{p}{\sqrt{(p+\rho)^2-\sigma^2}}\,1 = \epsilon^{-\rho t}J_0(j\sigma t)$ (492)

59. $\dfrac{\sqrt{p}}{p+b}\,1 = \dfrac{1}{b\sqrt{\pi t}}\left(1+\dfrac{1}{2\,bt}+\dfrac{1\cdot 3}{(2\,bt)^2}+\dfrac{1\cdot 3\cdot 5}{(2\,bt)^3}+\cdots\right)$

asymptotically (552)

60. $\dfrac{p}{p+b}\,\sqrt{p}\,1 = -\dfrac{1}{\sqrt{\pi t}}\left(\dfrac{1}{2\,bt}+\dfrac{1\cdot 3}{(2\,bt)^2}+\dfrac{1\cdot 3\cdot 5}{(2\,bt)^3}+\cdots\right)$

asymptotically (553)

61. $\dfrac{\sqrt{p}}{1+\sqrt{bp}}\,1 = \dfrac{1}{\sqrt{b}}\,\epsilon^{\frac{t}{b}} - \dfrac{2}{\sqrt{\pi b}}\sqrt{\dfrac{t}{b}}\left[1+\dfrac{2}{3}\left(\dfrac{t}{b}\right)+\dfrac{2^2}{3\cdot 5}\left(\dfrac{t}{b}\right)^2+\cdots\right]$ (561)

62. $\dfrac{\sqrt{p}}{1+\sqrt{bp}}\,1 = \dfrac{1}{\sqrt{\pi t}}\left(1-\dfrac{b}{2\,t}+\dfrac{3\,b^2}{2^2t^2}-\dfrac{3\cdot 5\,b^3}{2^3t^3}+\cdots\right)$

asymptotically (564)

63. $$\frac{1}{1+(bp)^{\frac{1}{2}}}1 = \frac{2^2}{\sqrt{\pi}\,3}\left(\frac{t}{b}\right)^{\frac{3}{2}} - \frac{1}{3!}\left(\frac{t}{b}\right)^3 + \frac{2^5}{\sqrt{\pi}\,3\cdot5\cdot7\cdot9}\left(\frac{t}{b}\right)^{\frac{9}{2}} - \frac{1}{6!}\left(\frac{t}{b}\right)^6 + \cdots \quad (574)$$

64. $$\frac{1}{1+(bp)^{\frac{1}{2}}}1 = 1 + \frac{1}{2\sqrt{\pi}}\left(\frac{b}{t}\right)^{\frac{1}{2}}\left[1 - 1\cdot3\cdot5\cdot7\left(\frac{b}{2t}\right)^3 + 1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\left(\frac{b}{2t}\right)^6 + \cdots\right] \text{asymptotically} \quad (584)$$

65. $$\epsilon^{-\sqrt{bp}}1 = 1 - \sqrt{\frac{b}{\pi}}\left(t^{-\frac{1}{2}} - \frac{1}{2}\frac{b}{3!}t^{-\frac{3}{2}} + \frac{1}{2}\frac{3}{2}\frac{b^2}{5!}t^{-\frac{5}{2}} - \cdots\right) \quad (588)$$

66. $$\epsilon^{-\sqrt{bp}}1 = 1 - erf\sqrt{\frac{b}{4t}} \quad (592)$$

67. $$\epsilon^{b(p-\sqrt{p^2-\sigma^2})}1 = 1 + \frac{b\sigma^2 t}{2} + \frac{b^2\sigma^4 t^2}{2^2 2!^2} + \left(\frac{b^3\sigma^6}{2^3 3!^2} + \frac{b\sigma^4}{2^2 2!3!}\right)t^3 + \left(\frac{b^4\sigma^8}{2^4 4!^2} + \frac{b^2\sigma^6}{2^3 2!4!}\right)t^4 + \cdots \quad (517)$$

68. $$\epsilon^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}1 = \epsilon^{-\rho t}\left[1 + \left(\frac{x\sigma^2}{2v} + \rho\right)\left(t - \frac{x}{v}\right) + \left(\frac{x^2\sigma^4}{2^2 2!^2 v^2} + \frac{x\sigma^2\rho}{2^2 v} + \frac{\rho^2}{2!}\right)\left(t - \frac{x}{v}\right)^2 + \cdots\right]1_{\frac{x}{v}} \quad (529)$$

69. $$\epsilon^{b(p-\sqrt{p^2-\sigma^2})}\frac{p}{\sqrt{p^2-\sigma^2}}1 = 1 + \frac{b\sigma^2 t}{2} + \left(\frac{b^2\sigma^4}{2^2 2!^2} + \frac{\sigma^2}{2\cdot2!}\right)t^2 + \left(\frac{b^3\sigma^6}{2^3 3!^2} + \frac{b\sigma^4}{2^2 2!^2}\right)t^3 + \left(\frac{b^4\sigma^8}{2^4 4!^2} + \frac{b^2\sigma^6}{2^5 2!^3} + \frac{\sigma^4}{2^4 2!^2}\right)t^4 + \cdots \quad (518)$$

70. $$\epsilon^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}\epsilon^{-\rho t}J_0(j\sigma t)\,1 = \epsilon^{-\rho t}J_0\left[j\sigma\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]1_{\frac{x}{v}} \quad (537)$$

71. $$\epsilon^{-b\sqrt{1+p^2}}\frac{p}{\sqrt{1+p^2}}1 = J_0(\sqrt{t^2-b^2})\,1_b$$

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